Recommended as a text-book for the Intermediate Examination by the University of Calcutta.

CONIC SECTIONS, CO-ORDINATE AND SOLID GEOMETRY

FOR INTERMEDIATE STUDENTS

WITH HINTS ON SOLUTION OF PROBLEMS.

BY

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THIRD EDITION

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PREFACE.

In presenting yet another text-book to the Intermediate students in mathematics we tender no apologies. Because, in the first place, our desire is to place at the hands of the students a clear and easy exposition of the subject, with numerous illustrative examples, either fully discussed or given sufficient hints to—so that, even the most average type of student can gain a complete mastery of the subject, to the extent desired, without feeling the strain of the work in the least. In the second place, we have taken full cognisance of the time-factor in the class-work in our colleges and have tried to present our brother teachers with a text that will enable them to save a lot of their valuable time without sacrificing an iota of efficiency. A recognition of the claims and merits of our work, if it is found to have any, will be our highest reward.

Inspite of our pretentious claims we admit we have been victims of speed. The rush and hurry for getting the book out in time has prevented us considerably from giving our best. We hope to make sufficient amends in the next edition.

Dated, Calcutta ;
The 7th July, 1939.

C. K. DHAR.
N. L. GHOSE.

PREFACE TO THE SECOND EDITION.

In this edition the book has been thoroughly revised. Some important additions and alterations have been made and recent University questions have been incorporated in the body of the book to acquaint the student with the trend of present-day examinations. It is hoped that the book in its present form will be found more useful to those for whom it is intended.

Any corrections or suggestions for the improvement of the work will be thankfully received.

Dated, Calcutta ;
The 1st Sept., 1943.

C. K. DHAR.
N. L. GHOSE.

PREFACE TO THE THIRD EDITION

The present edition is not merely a thorough revision of the previous one ; hints and solutions have been considerably abridged and the propositions have been presented in the usual order. New proofs have been added to one or two theorems. On the whole we expect the present edition to secure the satisfaction and approval of teachers and students in an added measure.

Dated, Calcutta ;
The 20th July, 1948.

C. K. DHAR.
N. L. GHOSE.

CALCULTTA UNIVERSITY SYLLABUS

FOR

THE INTERMEDIATE EXAMINATION

IN THE

GEOMETRY OF CONICS.

Parabola.

1. Tracing the curve from the definition.

2. Latus Rectum is four times the focal distance of the vertex.

3. $PN^2 = 4AS.AN$.

4. The middle points of parallel chords lie on a straight line parallel to the axis.

5. The parameter of any diameter of a parabola is four times the line joining the focus with the vertex of the diameter.

6. $QV^2 = 4BS.BV$.

7. If any chord QQ' intersects the directrix in D, SD bisects the exterior angle between SQ and SQ'.

8. The tangent to the curve at its point of intersection with a diameter is parallel to the system of chords bisected by the diameter.

9. The portion of the tangent at any point intercepted between that point and the directrix subtends a right angle at the focus.

10. The tangent bisects the angle between the focal distance and the perpendicular on the directrix.

11. The sub-tangent is bisected at the vertex.

Ellipse.

1. Tracing the curve from the definition.

2. The ellipse is symmetrical with respect to the minor axis, and has a second focus and directrix.

3. $CS.CX = CA^2$.

4. $SP + S'P = AA'$.

5. $CB^2 = SA.SA'$.

6. If any chord QQ' of an ellipse intersects the directrix in D, SD bisects the exterior angle between SQ and SQ'.

7. The middle points of parallel chords lie on a straight line passing through the centre.

8. The tangent to the curve at either end of a diameter is parallel to the system of chords bisected by the diameter.

9. The portion of the tangent at any point intercepted between that point and the directrix subtends a right angle at the focus, and conversely.

10. The tangents at the ends of a focal chord intersect on the directrix.

11. The tangent at any point of an ellipse makes equal angles with the focal distances of the point.

CALCUTTA UNIVERSITY SYLLABUS

FOR

THE INTERMEDIATE EXAMINATION

IN

ELEMENTS OF CO-ORDINATE GEOMETRY.

Finding out the equations of a straight line, circle, parabola and ellipse in their simplest forms from geometrical properties :

For Straight Line $\frac{x}{a} + \frac{y}{b} = 1.$

For Circle $x^2 + y^2 = a^2.$

For Parabola $y^2 = 4ax.$

For Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

CALCUTTA UNIVERSITY SYLLABUS

FOR

THE INTERMEDIATE EXAMINATION

IN

SOLID GEOMETRY.

1. One, and only one, plane may be made to pass through any two intersecting straight lines.

2. Two intersecting planes cut one another in a straight line and in no point outside it.

3. If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is also perpendicular to the plane in which they lie.

4. All straight lines drawn perpendicular to a given straight line at a given point are coplanar.

5. If two straight lines are parallel, and if one of them is perpendicular to a plane, then the other is also perpendicular to the same plane.

6. (i) Of all straight lines drawn from an external point to a plane, the perpendicular is the shortest.

(ii) Of obliques, drawn from the given point, those which cut the plane at equal distances from the foot of the perpendicular are equal.

7. The projection of a straight line on a plane is itself a straight line.

8. If a straight line is perpendicular to a plane, any plane passing through the perpendicular is also perpendicular to the given plane.

9. The definition of dihedral and solid angles.

10. The students will be expected to have an idea of the following solids :

Sphere, Right Circular Cylinder, Right Prism, Rectangular Parallelopiped, Right Circular Cone, Square and Triangular Pyramids.

11. Expressions (without proof) of the surfaces and volumes of the solids mentioned above.

CONIC SECTIONS, CO-ORDINATE AND SOLID GEOMETRY

FOR

INTERMEDIATE STUDENTS.

CONIC SECTIONS.

INTRODUCTION.

A CONIC is a curve traced out by a point which moves in a plane in such a manner that its distance from a certain fixed point in the plane bears a constant ratio to its distance from a certain fixed st. line in the same plane.

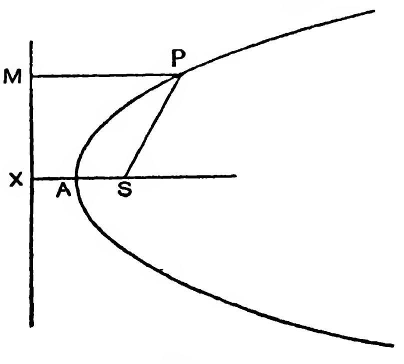

Fig. 1

In the above fig. MX is a fixed st. line and S is a fixed point ; if a point P moves in the plane determined by MX

and S in such a manner that its distance from S *viz.* SP bears a constant ratio to PM, its perp. distance from MX, then the locus of P is a conic. In other words, given MX and S, the locus of P is a conic, if $\dfrac{SP}{PM}=$ a constant.

In such a case, MX, the fixed line, is called the **Directrix** of the conic and S is called its **Focus.**

The value of the const. determines the nature of the conic.

If $\dfrac{SP}{PM}=1$, the conic is called a **Parabola.**

If $\dfrac{SP}{PM}<1$, the conic is called an **Ellipse.**

If $\dfrac{SP}{PM}>1$, the conic is called a **Hyperbola.**

The ratio $\dfrac{SP}{PM}$ is usually donoted by the letter "e" and is called the **Eccentricity** of the conic.

If SX is drawn perpendicular from S upon MX, we can find a point A in SX such that $\dfrac{SA}{AX}=e$, whatever be the value of e. Hence A must be a point on the conic determined by e. In other words, the conic cuts SX at the point A.

A is called the **Vertex** of the conic and the st. line SX its axis.

Thus, the **axis of a conic** *is the st. line through the focus perpendicular to the directrix.*

And, the **vertex** of a conic is its point of intersection with the axis.

The point of intersection of the axis with the directrix is generally denoted by X.

Note. Conics were originally obtained from the sections of **a right circular cone*** by a plane in different positions. Hence they are called **conic sections** or simply conics.

The different ways in which a plane may cut a rt. circular cone are the following :

(i) The plane may be parallel to one generator and perpendicular to the plane through that generator and the axis of the cone. The curve obtained in this case is a *Parabola.* Here the angle which the cutting plane makes with the axis of the cone is equal to the semi-vertical angle of the cone.

(ii) When the inclination of the plane to the axis is greater than the semi-vertical angle, the curve is closed and is an Ellipse.

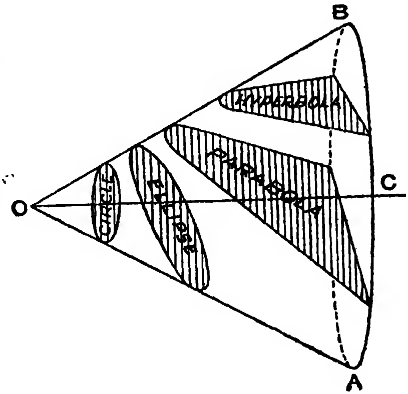

Fig. 2

(iii) When the inclination is less than the semi-vertical angle, the curve is a Hyperbola. In this case, the plane cuts both halves of the double cone and hence the hyperbola consists of two branches.

(iv) When the plane is perp. to the axis, the section is a Circle.

(v) When the plane passes through the axis, the section reduces to two intersecting st. lines.

The discovery of the conic sections is attributed to Menæchmus, a Greek mathematician who flourished in 350 B. C. nearly.

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*A right circular cone is the surface generated by the revolution of a right-angled triangle about one of its sides containing the right angle. In Fig. 2, AOB is the right circular cone of which OC is the axis and $\angle AOC$ or $\angle BOC$ is the semi-vertical angle of the cone.

CHAPTER I

THE PARABOLA.

CONSTRUCTION OF THE CURVE FROM THE DEFINITION.

PROPOSITION I.

Given the focus and the directrix of a parabola, to find any number of points on it ; i.e., to trace the curve from the definition.

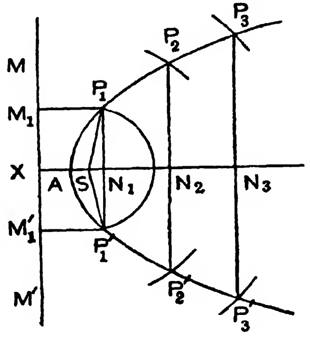

Fig. 3

Let S be the given focus and MM' the directrix of the parabola.

It is required to find any number of points on the parabola.

From S, the given focus, draw SX perpendicular to the directrix. Then SX is the axis of the parabola.

Bisect SX at A.

Then by definition, A is a point on the parabola and is called the **vertex.**

On AS or AS produced take any point N_1. Draw $P_1N_1P'_1$ perpendicular to the axis. With centre S and radius XN_1 describe a circle cutting $P_1N_1P'_1$ at P_1 and P'_1.

Then P_1 and P'_1 are points on the parabola.

Join SP_1, SP'_1.

Draw P_1M_1 and $P'_1M'_1$ perpendiculars to the directrix.

Since $XN_1 = M_1P_1$, and again $XN_1 = SP_1$.

$\therefore \quad M_1P_1 = SP_1$.

Similarly $SP'_1 = M'_1P'_1$.

Hence P_1, and P'_1 are points on the parabola.

Proceeding in a similar manner, and by taking points N_2, N_3 etc. on AS or AS produced, we can find any number of points on the parabola, and corresponding to each of the points N_3, N_2, etc., we get two points on it.

The curve drawn to pass through all these points will be the parabola required.

EXERCISES.

1. Having given a fixed point S and a fixed straight line MX, to trace the parabola by other methods.

First method : Join S to any point M on MX. Bisect MS at O; and draw OQ perpendicular to MS. Let the perpendicular MP to MX meet OQ at P. Then P is a point on the parabola, since $SP = PM$.

Second method : Draw MQ perpendicular to MX. Make an angle PSM equal to the angle SMQ and let SP meet MQ at P. Then P is a point on the parabola.

N. B. For another method of construction, see Prop. VIII, Ex. 9.

2. To construct a parabola mechanically. [C. U. 1910.]

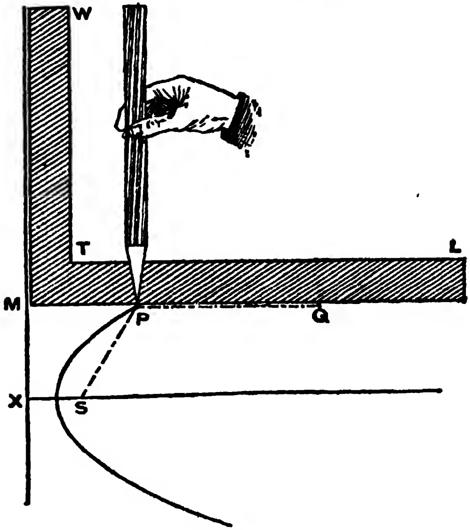

Fig. 4

Let S be the focus and MX the directrix of a parabola.

Then the perpendicular SX to the directrix is the axis.

Let the arm WT of a rigid framework, of which the arms WT and TL are rigidly connected to each other at right angles, slide along the directrix.

Take a fine string of length MQ and fasten one end at Q, a fixed point on the arm TL and the other end at S, the focus.

Now, let the arm WT slide along the directrix, so that it is always in contact with it, while the string is kept stretched by means of the point of a pencil at P, in contact with the bar.

The point of the pencil will describe a parabola, since SP is always equal to MP.

Note. If we draw a straight line mx parallel to MX through Q, then the sum of the distances of P from the straight line mx and the focus S is always constant and equal to MQ, the length of the string.

Hence we say that the locus of a point which moves in such a way that the sum of its distances from a fixed straight line and a fixed point is always a constant, is a parabola.

3. Given the directrix and two points on a parabola to trace the curve.

4. Having given the focus and two points on the curve, show how to describe it.

[Draw two circles with P, Q, the two points as centres and radii PS, QS respectively.]

5. Given the focus and the vertex of a parabola, show how to construct it.

Def. : *Any finite straight line joining two points on a conic is called a* **chord** of the conic.

Any chord of a conic passing through its focus is called a **focal chord** of the conic.

6. *Shew that the projection of a focal chord on the directrix subtends a right angle at the focus.*

[The $\angle MSm$ subtended by Mm, the projection of PSp on the directrix is half the straight angle PSp.]

Note. The *focal distance* of a point on a conic is its distance from the focus of the conic.

7. If P be any point on a parabola and SY be drawn perpendicular to SP meeting the directrix in Y, then PY bisects the angle between the focal distance and the perpendicular distance of P from directrix. Hence show that PY bisects SM at right-angles.

[Δs PMY, PSY are congruent.]

8. *The circle described on a focal chord of a parabola as diameter touches the directrix.* [C. U. 1917.]

Let PSp be the focal chord and V the middle point of Pp. Draw KV parallel to the axis meeting the directrix in K and PM, pm perpendiculars to the directrix. Join KP, Kp.

$$MP+mp=Pp, \text{ but from Geometry, } MP+mp=2KV.$$

$$\therefore \quad KV=VP=Vp.$$

$\therefore$ the circle described with Pp as diameter touches the directrix which is perpendicular to the radius KV.

9. If through the middle point V of a focal chord PSp, KV be drawn parallel to the axis meeting the directrix in K, shew that the focal chord Pp subtends a right angle at K. [*See Ex. 8.*]

10. *If PSp be a focal chord meeting the parabola in P and p and PN, pn are drawn at right angles to the axis, shew that* $AN.An = AS^2$.

From Δs SPN and Spn, because they are similar.

$$\frac{SN}{Sn} = \frac{SP}{Sp} = \frac{XN}{Xn} = \frac{XS+SN}{XS-Sn} = \frac{2AS+SN}{2AS-Sn}.$$

$$\therefore \quad \frac{SN}{Sn} = \frac{2AS+NS+SN}{2AS-Sn+Sn} = \frac{2AN}{2AS} = \frac{AN}{AS}.$$

Again, $$\frac{SN}{Sn} = \frac{2AS+NS-SN}{2AS-Sn-Sn} = \frac{2AS}{2An} = \frac{AS}{An}.$$

Thus $$\frac{SN}{Sn} = \frac{AN}{AS} = \frac{AS}{An}. \qquad \therefore \quad AN.An = AS^2.$$

11. *The distance of any point from the focus is greater than its distance from the directrix, if the point be outside the parabola.*

Let Q be any point outside the parabola.

Join QS meeting the parabola in P. Draw PM and QM' perpendiculars to the directrix.

Join MQ. P being a point on the parabola, $PS = PM$. Adding QP to each side, $MP+PQ = SQ$. But $MP+PQ > MQ$ and $MQ > M'Q$.

$$\therefore \quad SQ > M'Q.$$

12. *The distance of any point within a parabola from the focus is less than its distance from the directrix.*

13. *Show that a straight line parallel to the axis of a parabola meets it at one point and one point only. Hence shew that the parabola has only one vertex.* [C. U. 1923.]

Let KBC be any straight line parallel to the axis. If possible, let it meet the directrix in K and the curve in two points B and C.

Join BS, CS. Then $BS = BK$ and $SC = CK$;

$\therefore$ $CS - BS = CK - BK$; *i.e.*, $BC = SC - BS$, which is impossible, since BSC is a triangle. (Otherwise, compare the base angles of the $\triangle$s BKS, CKS).

If possible, let the axis meet the curve in two points A and A'.

Then $XA' = A'S$; *i.e.*, a part is equal to the whole, which is absurd.

14. *Shew that the parabola is not a closed curve.*

Any straight line meets a closed curve at least in two points, but a straight line parallel to the axis meets the parabola in one point only.

15. *The locus of the centre of a circle which touches a given circle and a given straight line is a parabola.*

The focus is the centre S of the given circle and the directrix a straight line parallel to the given line on the side of it remote from S at a distance equal to the radius of the circle.

16. *To find the locus of the centre of a circle which passes through a given point and touches a given straight line.*

The locus is a parabola with the given point as the focus and the given straight line as the directrix.

17. *If any chord PSp of a parabola is produced to meet the directrix at Y, prove that $Sp.PY = PS.Yp$.*

[Draw PM and pm perpendiculars to the directrix. The triangles PMY and pmY are similar.]

18. *Show that the perpendicular from the middle point of a focal chord to the directrix is half the focal chord.* [*See Ex. 8*]

19. If PSp is a focal chord of a parabola, shew that

$$\frac{1}{SP} + \frac{1}{Sp} = \frac{1}{AS}.$$

[From Ex. 10, $AN.An = AS^2$; hence $\dfrac{AS}{AN} = \dfrac{An}{AS}$.

$$\therefore \quad \frac{AS}{AS+AN}=\frac{An}{AS+An} \quad \text{or} \quad \frac{AS}{XN}=\frac{An+AS-AS}{An+AS},$$

$$\text{or} \quad \frac{AS}{SP}=1-\frac{AS}{AS+Sn}=1-\frac{AS}{Xn}=1-\frac{AS}{Sp}.$$

$$\text{Hence} \quad \frac{AS}{SP}+\frac{AS}{Sp}=1 \quad \text{or} \quad \frac{1}{SP}+\frac{1}{Sp}=\frac{1}{AS}.\ ]$$

SYMMETRY.

Def. : *A curve is said to be* **symmetrical about** *a straight line when any chord of the curve perpendicular to the straight line is bisected by it.*

PROPOSITION II.

*To show that the parabola is symmetrical with respect to its axis.**

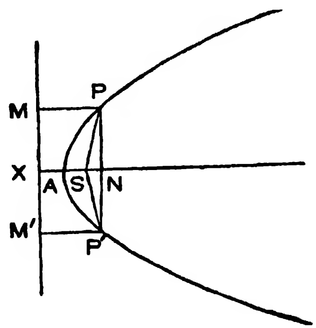

Fig. 5

Let S be the focus and MXM' the directrix of a parabola.

Let PNP' be any chord of a parabola, meeting it in P and P' and perpendicular to the axis XAS.

* C. U. 1912, '23.

It is required to prove that $PN = P'N$.

Join SP, SP'. Draw PM and $P'M'$ perpendicular to the directrix.

In the Δs SPN, $SP'N$, we have $SP = SP'$,

[$\because$ $SP = MP = XN$ and $SP' = M'P' = XN$.]

SN common and the Δs are right-angled Δs ;

$\therefore$ the two triangles are congruent.

Hence $PN = P'N$.

But PNP' is any chord at right angles to the axis ;

$\therefore$ all chords perpendicular to the axis are bisected by it.

Hence the parabola is symmetrical with respect to its axis.

Note 1. An axis of a curve is a straight line about which the curve is symmetrical.

Note 2. If the parabola be folded about the axis then the portion of the parabola on one side of the axis coincides completely with the portion on the other side. Hence a curve is said to be symmetrical with respect to a straight line, when on being folded, the portion of the curve on one side of it coincides completely with the portion on the other. This is another definition of symmetry.

Note 3. Two points on a parabola are said to be corresponding points, if one point falls on the other, when the parabola is folded about the axis. Here the points P and P' are said to be two corresponding points on the parabola.

EXERCISES.

1. If two straight lines are drawn from the focus of a parabola, equally inclined to the axis and meeting the curve in P and P', shew that PP' is bisected at right angles by the axis at N. [Fold the parabola about the axis. Prove that Δs SPN, $SP'N$ are congruent.]

2. *If through any point on the axis two chords are drawn on opposite sides of the axis making equal angles with it, shew that the two chords are equal in length and conversely.* [Fold the parabola about its axis as in Ex. 1. Follow the indirect method for the converse proof.]

3. If through any point O on the axis, two straight lines OP and OP' are drawn one on each side of it and equally inclined to it, shew that any point N on the axis is equidistant from P and P'.

4. *Prove that a parabola lies on one side of the directrix only and it extends infinitely on that side.* [C. U. 1912.]

Let S be the focus, MXM' the directrix of a parabola, and N any point on the axis. The construction fails if $SN > XN$, which is the case, when N is somewhere on AX or AX produced.

5. *Given a chord perpendicular to the axis, find another parallel chord, which shall be double the given chord.*

6. Given a parabola, find its axis. [Fold it symmetrically.]

THE LATUS RECTUM.

Def. : *The latus rectum of a parabola is the focal chord perpendicular to the axis of the parabola.*

PROPOSITION III.

*The latus rectum of a parabola is four times the focal distance of the vertex.**

Let S be the focus, MXM' the directrix and A the vertex of the parabola. Join SA and produce it to meet MXM' at X. Then SX is the axis and is perp. to the directrix.

Through S, draw a focal chord LSL' perpendicular to the axis. Then LSL' is the latus rectum.

It is required to prove that $LL' = 4AS$.

* C. U. 1909, '15, '20, '25.

Draw LM and $L'M'$ perpendiculars to the directrix meeting it in M and M' respectively.

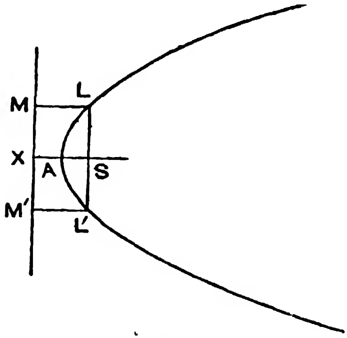

Fig. 6

Since L is a point on the parabola, $LS = LM$.

Again, LL' being perpendicular to the axis, $ML = XS$.

Hence the rectangle $MLXS$ is a square, so that

$$LS = XS.$$

$$\text{But } XA = AS, \quad \therefore\ LS = 2AS.$$

$$\text{Similarly, } SL' = 2AS.$$

$$\text{Hence, } LL' = 4AS.$$

Definitions.

The **ordinate** *of any point on a parabola is the perpendicular drawn from the point on the axis. It is also called the* **principal ordinate** *of the point.*

Any chord of a parabola which is perpendicular to the axis is called a **double ordinate** *of a parabola.*

If PP' be a chord of a parabola perpendicular to the axis, then PP' is called a double ordinate.

EXERCISES.

1. Shew that the circle described on the latus rectum as diameter touches the directrix at the point at which the axis meets it.

2. *To find the ordinate of a parabola which is equal to the latus rectum.* [C. U. 1920]

3. *Find a double ordinate of a parabola which shall be double the latus rectum.* [C. U. 1909.]

4. If $4a$ be the latus rectum of a parabola, find the radius of the circumcircle of the $\triangle LAL'$.

[Let AD be the diameter. Then $\triangle ALD$ is rt.-angled. Hence, Diameter $\times AS = AL^2 = 5AS^2$.]

5. Shew that $AL^2 = \frac{5}{16} L'L^2$; *i.e.*, $\frac{5}{16}$(latus rectum)^2.

6. If the ordinate PN of a parabola be produced outwards to Q to meet AL produced (the line joining the vertex to the end of the latus rectum), shew that $QN = 2AN$.

7. *Given a parabola and its axis in position, find the focus and the directrix.* [C. U. 1915, '25]

[At any point N on the axis, draw QN perpendicular to the axis making $QN = 2AN$. Join QA cutting the parabola in L.]

8. To find the point Q in a given ordinate PN, such that if Qy be drawn parallel to the axis to meet the curve in y, $QN + Qy$ is the greatest.

[Draw a straight line parallel to the axis, through the end of the latus rectum, cutting PN at Q, which is the required point.
Then, $QN + Qy = XN$. For any other point it is less than XN.]

9. *The rectangle contained by the projections of the sections of a focal chord by the axis on the directrix is a constant* $= \frac{1}{4}$ *(latus rectum)*^{2}.

If MXm be the projection of the focal chord PSp, then the $\triangle$s MSX, mSX are similar. [*Ex. 6, Prop. I.*]

$$\therefore\ MX.mX = SX^2.$$

The Relation between the Ordinate and the Abscissa of a Point.

Definition. *The* **abscissa** *of any point on a parabola is the portion of the axis intercepted between the vertex and the ordinate of the point.*

Proposition IV.

*The square of the ordinate of any point on a parabola is equal to the rectangle contained by the latus rectum and the abscissa of the point.**

$$(PN^2 = 4AS.AN.)$$

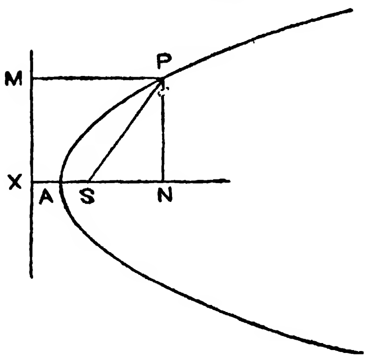

Fig. 7

Let S be the focus and MX the directrix of any parabola.

Draw SX perpendicular to the directrix, cutting the parabola in A and the directrix in X.

Then XAS is the axis of the parabola.

From any point P on the parabola, draw PN perpendicular to the axis, cutting it at N, so that PN is the ordinate of the point P and AN the abscissa.

* C. U. 1913, '18, '24, '30, '33, '35, '37.

It is required to prove that $PN^2 = 4AS.AN$.

Join SP.

Draw PM perpendicular to the directrix.

PSN is a right-angled triangle,

$\therefore \quad SP^2 = PN^2 + SN^2,$

i.e., $PN^2 = SP^2 - SN^2$

$= XN^2 - SN^2$* $[\because \; SP = MP = XN.]$

$= (XN + SN)(XN - SN)$

$= (XS + 2SN)(XS)$

$= (2AS + 2SN)\, 2AS \quad [\because \; XA = AS.]$

$= 4AS\,(AS + SN) = 4AS.AN.$

Note 1. $PN^2 = 4AS.AN$. For a given parabola since the latus rectum $4AS$ is a constant, $PN^2 \propto AN$. Hence the theorem may be enunciated thus :—

The square of the ordinate of any point of a parabola varies as the abscissa, the constant of variation being the latus rectum.

Note 2. $PN^2 = 4AS.AN$; $\therefore\ 4AS : PN :: PN : AN$. Hence the proposition may be enunciated in another form.

The ordinate of any point of a parabola is the mean proportional between the abscissa of the point and the latus rectum.

EXERCISES.

1. If a point P moves in a plane in such a way that the square of the perpendicular distance of P from a fixed straight line passing through a given point A varies as the distance of the foot of the perpendicular from the given point A, shew that the locus of P is a parabola. [Prove it by the indirect method.]

The fixed point is the vertex, and the fixed straight line is the axis of the parabola.

2. Find the locus of points which divide the ordinates of a parabola in a given ratio. Deduce the case when the ratio is one. [C. U. 1924.]

*$PN^2 = XN^2 - SN^2 = (AN + AS)^2 - (AN - AS)^2 = 4AS.AN.$

$(\because \; AX = AS.)$

[Let PN be any ordinate of a parabola, which is divided at Q in the ratio $m : n$.

$$\therefore \quad \frac{PQ}{QN}=\frac{m}{n}. \quad \therefore \quad \frac{PQ+QN}{QN}=\frac{m+n}{n}.$$

$$\therefore \quad QN=\frac{n}{m+n}PN. \quad \therefore \quad QN^2=\left(\frac{n}{m+n}\right)^2 PN^2$$

$$i.e., \quad QN^2=\left(\frac{n}{m+n}\right)^2 4AS.AN=k.AN,$$

where k (a constant) $=4AS.\left(\frac{n}{m+n}\right)^2$.]

The locus of Q is a parabola, having the same vertex and axis, the latus rectum being equal to k. [*Ex. 1.*] When $m=n=1$, $k=AS$.

3. The latus rectum is mean proportional between the double ordinates of the extremities of a focal chord. [C. U. 1937.]

[Let PSp be a focal chord and PN and pn, the ordinates of P and p respectively.

$$pn^2=4AS.An. \quad PN^2=4AS.AN.$$

$$\therefore \quad 2pn.2PN=4PN.pn=16AS.\sqrt{AN.An}=16AS^2.$$

[*Prop. I, Ex. 10*]

4. Prove that the rectangle contained by the abscissæ of two points P and P' on a parabola, such that PAP' is a right angle, is equal to $16AS^2$.

[Draw the ordinates PN and $P'N'$. The $\triangle$s $AP'N'$, APN are similar, so that $PN.P'N'=AN.AN'$.]

5. Find the ordinate of a parabola, which is equal to the abscissa. [The required ordinate is equal to the latus rectum.] [C. U.]

6. The common chord of a parabola and a circle with centre the vertex of the parabola and radius $\frac{3AS}{2}$ bisects AS.

[If PNP' be the common chord of the circle and the parabola, cutting the axis at N, then $AP^2=PN^2+AN^2=4AS.AN+AN^2$.

Put $AP=\frac{3AS}{2}$ and find AN.]

7. *If PQ be drawn at right angles to any chord AP of a parabola to meet the axis at Q, shew that QN is equal to the latus rectum, where N is the foot of the ordinate of P.* [C. U. 1918, '30, '40.]

[The $\triangle$s APN, PQN are similar, so that $PN^2=QN.AN$.]

8. The latus rectum intersects the sides AQ and Aq of the $\triangle AQq$, where Qq is any focal chord, at points whose focal distances are equal to the ordinates of the focal chord.

[Let AQ, Aq meet the latus rectum at O and O' respectively. Draw QN and qn principal ordinates to Q and q respectively.

The $\triangle$s ASO, ANQ are similar.

$$\therefore \frac{OS}{QN}=\frac{AS}{AN}. \quad \therefore OS=\frac{AS.QN}{AN};$$

$$\therefore \quad OS^2=\frac{AS^2.QN^2}{AN^2}=\frac{AS^2.4AS.AN}{AN^2}=\frac{4AS.AS^2}{AN}=\frac{4AS.AN.An}{AN}$$

$$=4AS.An=qn^2. \qquad [\textit{ Prop. I, Ex. 10. }]$$

9. Find the length of the tangent drawn from A to the circle circumscribed about the $\triangle SPN$.

$$\left[\text{(Length to the tangent)}^2=AS.AN=\frac{PN^2}{4}.\right]$$

10. Given a parabola to find the focus and the directrix.

11. A straight line AY is drawn perpendicular to the axis of a parabola, and another line MQ is drawn parallel to the axis through the middle point M of PN, cutting the curve in Q. If NQ produced meet AY in Y, show that $PN=\frac{3}{2}AY$.

[Draw Qn ordinate of Q. $\triangle$s OYN, QNn are similar, so that

$$\left.\frac{AY}{Qn}=\frac{AN}{Nn}=\frac{AN}{AN-An}=\frac{4AS.AN}{4AS(AN-An)}=\frac{PN^2}{PN^2-\frac{1}{4}PN^2}=\frac{4}{3}\ \right]$$

12. If QNq be a double ordinate of a parabola, such that a chord AP and a straight line PM parallel to the axis meet it in M', M respectively, then $QN^2=NM.NM'$.

[The $\triangle$s PAn, $AM'N$ being similar, where Pn is the ordinate of P,

$$\frac{Pn}{NM'}=\frac{AN}{An}, \quad \text{or,} \quad \frac{Pn^2}{NM'.NM}=\frac{An}{AN}, \text{ since } Pn=MN.]$$

13. Ordinates P_1N_1, P_2N_2, P_3N_3 to a parabola are in the ratio of 1 : 2 : 3. Shew that AN_1, N_1N_2, N_2N_3 are in the ratio of 1 : 3 : 5.

[C. U. 1913.]

[$AN_1 : AN_2 : AN_3=PN_1^2 : PN_2^2 : PN_3^2=1 : 4 : 9.$]

14. Shew that the locus of the middle points of all focal chords of a parabola is a similar parabola through the focus.

[**Hints** : Let $p \to$ mid-point of a chord QSQ' and pn, ordinate of p. Then $pn=\frac{1}{2}(QN-Q'N')$ and $An=\frac{1}{2}(AN+AN')$. $\therefore$ $pn^2=2AS.Sn$.]

PROPOSITION V.

*The locus of the middle points of any system of parallel chords of a parabola is a straight line parallel to the axis.**

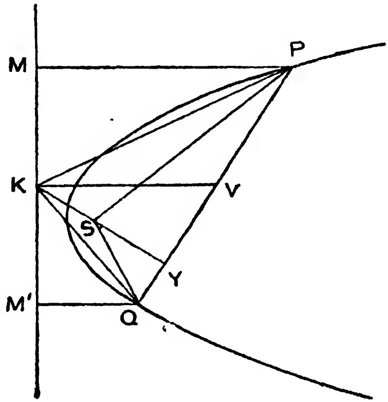

Fig. 8

Let PQ be one of a system of parallel chords of a parabola of which S is the focus, MM' the directrix.

It is required to prove that the middle points of the system of chords parallel to PQ lie in a straight line parallel to the axis.

Draw SY perpendicular to PQ and produce it to meet the directrix in K.

Draw PM and QM' perpendiculars on the directrix and KV parallel to the axis meeting PQ in V.

Then KV shall bisect the system of parallel chords.

Join KP, KQ, SP and SQ.

Then $MK^2 = KP^2 - MP^2 = KP^2 - SP^2 \cdots\cdots$ (1)

*C. U. 1910, '15, '21, '26, '34, '38, '41.

But $KP^2 = KY^2 + YP^2$, and $SP^2 = SY^2 + YP^2$.

$\therefore \quad KP^2 - SP^2 = KY^2 - SY^2 \cdots \qquad \cdots \quad (2)$

$\therefore$ from (1) and (2), $MK^2 = KY^2 - SY^2. \qquad \cdots \quad (3)$

Similarly, $M'K^2 = KQ^2 - M'Q^2 = KQ^2 - SQ^2$

$= KY^2 - SY^2. \qquad \cdots \qquad \cdots \quad (4)$

[$\because KQ^2 = KY^2 + QY^2$ and $SQ^2 = QY^2 + SY^2$.]

From (3) and (4), $MK = M'K$.

Thus, MP, KV and MQ are parallel straight lines cutting MM' and PQ and $MK = M'K$.

$\therefore$ by the well known proposition in Geometry,

$PV = QV$, *i. e.* KV bisects PQ.

Now, as KY is perpendicular to PQ, it is perpendicular to all straight lines parallel to PQ. Hence, as PQ is fixed in direction KY is also fixed in direction and further KY passes through S. That is, KY is a fixed line for all parallel chords and hence K is a fixed point. Again KV, being drawn parallel to the axis, is also a fixed straight line.

Thus the middle points of all chords parallel to PQ will lie on KV.

Hence, the middle points of any system of parallel chords of a parabola is a st. line parallel to the axis.

Definitions.

A **diameter** *of a conic is the locus of the middle points of a system of parallel chords of the conic.*

The **vertex** *of a diameter of a parabola is the point of intersection of the diameter with the curve. It is generally denoted by the letter B.*

If a chord be drawn from a point on a parabola so as to be bisected by a given diameter, then the semi-chord intercepted between the curve and the diameter is called the **ordinate** *of the point with respect to the diameter.*

The **abscissa** *of the point with respect to the diameter is the portion of the diameter intercepted between the ordinate and the curve.*

The axis of a parabola is the diameter bisecting all chords perpendicular to itself. It is the *principal diameter* of the parabola.

The term 'ordinate of a point on a parabola' is not definite, unless we mention the diameter with respect to which the ordinate is to be taken. As there is an infinite number of diameters of a parabola, so there is an infinite number of ordinates of a point.

Similarly the abscissa of a point is not definite so long as we do not mention the diameter.

On the other hand the term 'principal ordinate' of a point is always definite, because principal ordinate is always taken with respect to the axis of a parabola.

Conventionally however, 'the ordinate of a point on a parabola,' simply, means the ordinate with respect to the axis.

Note. Thus, in the figure of Proposition V, the half-chords PV and QV are the ordinates of P and Q with respect to the diameter KV.

If the point at which KV cuts the curve be denoted by B, then BV is called the abscissa of the point P with respect to the diameter KV.

EXERCISES.

1. Diameters of a parabola are perpendicular to the directrix.

2. The middle points of any two chords of a parabola equally inclined to the axis are equidistant from it.

3. A parabola being traced on a piece of paper, find the focus, the directrix and the axis. [C. U. 1934.]

[Draw any two parallel chords and join their middle points by a straight line. Take any other chord PQ perpendicular to this straight line. Through the middle point of this chord, draw a straight line perpendicular to PQ, which is the axis of the parabola.]

[*See Ex.* 7, *Prop. III*]

4. *The difference between the segments of any chord of a parabola, made by the axis is equal to the parallel chord through the vertex.*

[C. U. 1941.]

5. A system of parallel chords of a parabola is inclined at an angle θ to the axis. Find the distance of their diameter from the axis. Calculate the value of θ, when the diameter passes through an extremity of the latus rectum.

6. *If PSp be a focal chord and PN, pn be the ordinates with respect to the axis, Nn is equal to the parallel chord through the vertex.*

[Difference of the segments $= PS - pS = XN - Xn = Nn$. (*See Ex* 4.)]

7. Draw a chord of a parabola, passing through a given point and bisected by a given diameter.

[If K be the point of intersection of the given diameter and the directrix, join K with the focus S. Draw a focal chord perpendicular to KS. The chord drawn through the given point and parallel to the focal chord is the required one.]

8. *The diameter of a system of parallel chords and the focal perpendicular on them meet on the directrix.* [C. U. 1910, '21.]

[From K, the point of intersection of the focal perpendicular with the directrix, draw a straight line parallel to the axis, and prove it to be the diameter.]

Note. The focal perpendicular on a chord is defined to be the perpendicular from the focus on the chord.

9. A focal chord PSp of a parabola subtends a right-angle at the point K at which the diameter KBV bisecting it, intersects the directrix. [See Ex. 8 page 7.]

10. A chord perpendicular to KS is bisected by the diameter through K, where K is any point on the directrix.

11. AQ is any chord through the vertex, and QL is drawn perpendicular to it, meeting the axis in L ; shew that the length of the focal chord parallel to AQ is equal to AL. [C. U. 1938, '45.]

[Let PSp be the focal chord parallel to AQ.

Draw PN, pn perpendiculars on the axis.

The $\triangle$s PSN, pSn and AQL are similar.

$\therefore \dfrac{AL}{AQ}=\dfrac{PS}{SN}=\dfrac{pS}{Sn}=\dfrac{Pp}{Nn}$. But $AQ=Nn$. [*Ex.* 6.]

$\therefore Pp=AL$.]

THE PARAMETER.

Def. : *The* **parameter** *of any diameter of a parabola is the length of the focal chord bisected by that diameter.*

The latus rectum is the parameter of the axis.

Proposition VI.

*The parameter of any diameter of a parabola is four times the line joining the focus with the vertex of the diameter.**

[$Pp = 4BS.$]

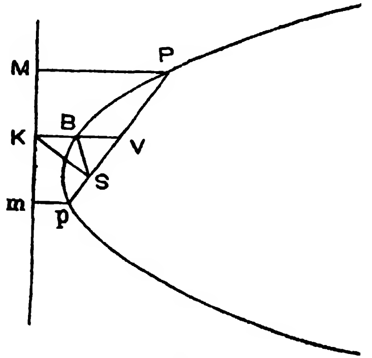

Fig. 9

Let PSp be a focal chord of a parabola of which S is the focus and Mm the directrix. Draw KS perpendicular to PSp, meeting the directrix in K. Let KV be perpendicular to the directrix meeting the parabola in B and PSp in V. Then KV is the diameter of the focal chord PSp, so that the latter is the parameter of the diameter KV and B is the vertex of KV. [*Prop.* V.]

Join BS.

It is required to prove that $Pp = 4BS$.

Draw PM and Pm perpendiculars on the directrix.

Then $Pp = PS + Sp = MP + mp = 2KV$.† ... (1)

* C. U. 1911, '14, '22, '29, '36, '39, '42.

† It may be proved geometrically as follows :

Join Mp meeting KV in O.

$\therefore$ $KO = \frac{1}{2}mp$, and $OV = \frac{1}{2}Mp$. $\therefore$ $Mp + mp = 2(KO + OV) = 2KV$.

[$\because$ MP, KV and mp are parallels and K is the middle point of Mm.]

Again the $\angle BKS$ = the $\angle BSK$.

[$\because$ from the definition of a parabola, $KB = BS$.]

Now, $\angle BKS = 90° - \angle BVS$. $\because$ $\angle KSV$ is a rt. $\angle$,

and $\angle BSK = 90° - \angle BSV$ for the same reason.

$\therefore$ the $\angle BSV$ = the $\angle BVS$.

Hence $BS = BV$, but $KB = BS$. $\therefore$ $KV = 2BS$.

$\therefore$ from (1), $Pp = 2KV = 4BS$.

EXERCISES.

1. *Draw a focal chord of a parabola which shall be double a given straight line.* (*Draw a focal chord of given length.*) [C. U. 1936]

2. *Of all focal chords the latus rectum is the least.*

[C. U. 1914, '22, '29.]

3. *Draw a focal chord PSp of a parabola, so that*

$\frac{SP}{Sp} = 3$ Generalise the case, when $\frac{SP}{Sp} = n$. [C. U. 1911, '39.]

[A focal chord drawn at an angle 60° to the axis is divided in the ratio of 1 : 3 by the axis.

In the figure of Proposition VI,

when the $\angle KVS = 60°$, the $\angle VKS = 30°$.

Consequently $KV = 2SV$, and $PV = KV$.

$$\therefore \; SP = PV + VS = KV + \tfrac{1}{2}KV = \tfrac{3}{2}KV,$$

$$\text{and} \quad Sp = pV - SV = KV - \tfrac{1}{2}KV = \tfrac{1}{2}KV.$$

In order to generalise the proposition, we proceed thus. Let SP be of length $2nk$, and Sp be of length $2k$, such that $SP : Sp = n : 1$. Then $KV = \frac{1}{2}(SP + Sp) = (n+1)k$; and $SV = pV - Sp = (n+1)k - 2k = (n-1)k$.

Hence deduce $KX = \frac{n-1}{\sqrt{n}} \cdot AS$.]

4. *PSp is the focal chord of a parabola with the focus at S, bisected by the diameter KBV, K being its point of intersection with the directrix. Show that $KS^2 = SP.Sp$.* [C. U. 1942.]

[*See Prop. V, Ex. 9*]

Hints. $KV = \frac{1}{2}(PS + ps)$, $SV = \frac{1}{2}(PS - ps)$.

Relation between the Ordinate and the Abscissa with regard to any Diameter.

Proposition VII.

*The square of the ordinate to any diameter of a parabola at any point is equal to the rectangle contained by its parameter and the abscissa of the point with respect to the diameter** [$QV^2 = 4BS.BV.$]

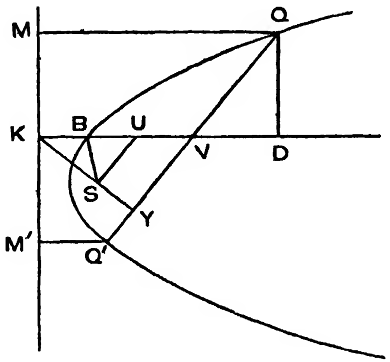

Fig. 10

Let QQ' be any chord of a parabola of which S is the focus and MM' the directrix. Draw SY perpendicular to QQ', meeting the directrix in K; and draw KV parallel to the axis of the parabola, meeting the curve in B and the chord QQ' in V. Then QV is the ordinate and BV the abscissa of the point Q with respect to the diameter BV. Join BS.

It is required to prove that $QV^2 = 4BS.BV.$

Draw QM, $Q'M'$ perpendiculars on the directrix, meeting it in M and M'. Draw SU parallel to the chord

* C. U. 1910, '16, '27.

QQ', meeting the diameter BV in U, and QD perpendicular to the diameter meeting it at D.

The right-angled Δs QVD, KYV are similar.

$$\therefore \quad \frac{QV}{QD}=\frac{KV}{KY} \quad \cdots \quad \cdots \quad (1)$$

Since in the ΔKYV, SU is parallel to YV,

$$\therefore \quad \frac{KV}{VU}=\frac{KY}{SY}. \quad \therefore \quad \frac{KV}{KY}=\frac{VU}{SY} \cdots \text{(Alternando)} \quad (2)$$

From (1) and (2), $\dfrac{QV}{QD}=\dfrac{KV}{KY}=\dfrac{VU}{SY}$.

$$\therefore \quad \frac{QV^2}{QD^2}=\frac{KV^2}{KY^2}=\frac{VU^2}{SY^2}. \therefore \quad \frac{QV^2}{QD^2}=\frac{KV^2-UV^2}{KY^2-SY^2}* \quad \cdots \quad (3)$$

But it has already been shown that $MK^2=KY^2-SY^2$.

[*Prop. V.*]

$$\therefore \quad QD^2=MK^2=KY^2-SY^2.$$

$$\therefore \quad \text{from (3)}, \ QV^2=KV^2-VU^2. \quad \cdots \quad (4)$$

But KU, the hypotenuse of the right-angled ΔKSU is bisected at B. [*Prop. VI.*]

$$\therefore \quad KB=BU=BS.$$

Hence $\quad KV=BV+BS$ and $VU=BV-BS$,

$$\therefore \quad QV^2=(BV+BS)^2-(BV-BS)^2, \text{ from (4)}$$
$$=4BS.BV.$$

Note. **The Proposition IV is a particular case of Proposition VII; when the diameter KV coincides with the axis of the parabola, the oblique ordinate QV and the abscissa BV become the principal ordinate and the principal abscissa of Q, and BS becomes AS.**

$$*\frac{KV^2}{KY^2}=\frac{VU^2}{SY^2}. \qquad \frac{KV^2}{VU^2}=\frac{KY^2}{SY^2}.$$

$\therefore$ **subtracting one from each side,** $\dfrac{KV^2-VU^2}{VU^2}=\dfrac{KY^2-SY^2}{SY^2}$.

$$\therefore \quad \frac{KV^2-VU^2}{KY^2-SY^2}=\frac{VU^2}{SY^2}=\frac{QV^2}{QD^2}.$$

The proposition may be enunciated as follows :

The ordinate of any point on a parabola to any diameter is mean proportional between the abscissa of the point and the parameter of the diameter.

EXERCISES.

1. If QQ' be any chord of a parabola having KV for the diameter of the system of chords parallel to it, shew that $QD^2 = 4AS.BV$, where QD is the perpendicular from Q on the diameter and B the vertex of the diameter. [C. U. 1947]

[In the figure of Proposition VII, draw SX perpendicular to the directrix and QD perpendicular to BV produced.

The Δs QVD, KUS are similar, and the Δs KSX, KSU are similar, whence $\dfrac{QD}{QV} = \dfrac{KS}{KU} = \dfrac{SX}{KS}$. $\therefore \left(\dfrac{QD}{QV}\right)^2 = \dfrac{KS}{KU} \cdot \dfrac{SX}{KS} = \dfrac{SX}{KU} = \dfrac{2AS}{2BS} = \dfrac{AS}{BS}$.]

2. PQ is any chord of a parabola, and BOC is any diameter meeting it in O. Shew that BO is mean proportional between the abscissæ of P and Q with respect to BOC.

[Let PV, QV' be ordinates to the diameter BOC. Then Δs POV, QOV' are similar. Hence $\dfrac{QV'^2}{PV^2} = \dfrac{OV'^2}{OV^2}$. $\therefore \dfrac{BV'}{BV} = \dfrac{OV'^2}{OV^2} = \dfrac{(BV' - BO)^2}{(BO - BV)^2}$.

Hence deduce $BO^2 = BV.BV'$.]

3. Find the ordinate of a parabola which will be equal to its abscissa. [$QV = BV = 4BS$.]

4. If through any point O on a diameter of a parabola two chords POQ, $P'OQ'$ be drawn, only the former being bisected by the diameter, PO is mean proportional between the ordinates of P' and Q' to the given diameter. [*Apply Ex. 2.*]

5. Draw a chord of given length, belonging to a system of parallel chords of a parabola with focus S, having given the vertex of the diameter to the parallel chords.

6. Find the locus of a point dividing any ordinate to a diameter of a parabola in a given ratio.

The locus is a parabola having that diameter as a diameter.

[*See Prop. IV, Ex. 2.*

7. *If PNQ be one of the system of parallel chords of a parabola, meeting its axis in N, shew that* $PN.QN \propto AN$.

[Through the middle point V of PQ, draw the diameter BV and through the vertex A, draw AV' parallel to PQ, meeting the diameter in V'.

Then $PN.\ QN = PV^2 - NV^2 = 4BS.BV - 4BS.\ BV = 4BS.\ AN$]

8. *Draw a chord through a given point so that it is divided in the ratio* $\frac{n}{m}$ *at the point.*

[Through the given point O, draw the diameter BO. If POQ be the chord to be drawn, let PV and QV' be ordinates to the diameter.

The $\triangle$s PVO and $QV'O$ being similar,

$$\frac{PV}{QV'} = \frac{PO}{QO} = \frac{n}{m}. \quad \therefore \left(\frac{PV}{QV'}\right)^2 = \left(\frac{BV}{BV'}\right) = \left(\frac{n}{m}\right)^2.$$

Again, $BV.BV' = BO^2$. [*Ex. 2.*] Thus V and V' are known points.]

The Secant.

Definition. *A* **Secant** *of a conic is a straight line that cuts it in two points.*

Note. A secant of a conic may therefore be defined to be a chord of indefinite length.

Proposition VIII.

If any chord QQ' of a parabola intersect the directrix in D, SD bisects the exterior angle between SQ and SQ'.

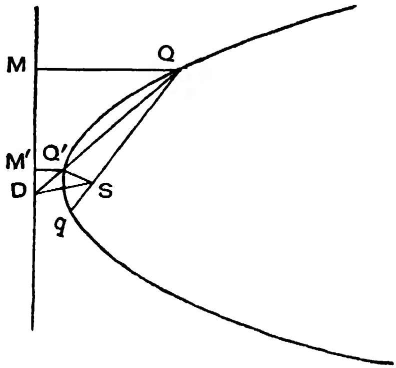

Fig. 11

Let QQ' be any secant of a parabola of which S is the focus, meeting the directrix in D.

Join QS and produce it to any point q. Join $Q'S$, DS.

It is required to prove that DS bisects the exterior angle $Q'Sq$ of the $\triangle QSQ'$.

Draw QM, $Q'M'$ perpendicular on the directrix.

Since QM and $Q'M'$ are parallel, the $\triangle$s DQM, $DQ'M'$ are similar.

$$\therefore \quad \frac{QD}{Q'D} = \frac{MQ}{M'Q'}.$$

But $\qquad QS = MQ$ and $Q'S = M'Q'$.

$$\therefore \quad \frac{QD}{Q'D} = \frac{QS}{Q'S}.$$

$\therefore$ the side QQ' of the $\triangle SQQ'$ is divided externally at D in the rate of the sides SQ and SQ'.

$\therefore$ from Geometry, SD bisects the exterior angle $Q'Sq$.

EXERCISES.

1. Pp and Qq are any two focal chords of ap arabola. Shew that PQ and pq intersect on the directrix. Also Pq and pQ intersect on the directrix.

2. *The portion of the directrix of a parabola intercepted between the straight lines joining the ends of a focal chord to any point on the parabola, subtends a right angle at the focus.*

[If Q be any point on the parabola and PSp any focal chord, let PQ, pQ meet the directrix at M and m. Then mS and MS are bisectors of the angles PSQ and pSQ.]

3. *If AP, any secant of a parabola passing through the vertex, meet the directrix at P', then the projection of PP' on the directrix subtends a right angle at the focus.*

4. Having given the focus and two points on a parabola, determine any number of points on it. Find the directrix.

5. *PSP' is a focal chord. The ends P, P' are joined to any variable point Q, where QP and QP' meet the directrix in p and p'. Prove that the rectangle contained by pX and $p'X$ is a constant* $(=4AS^2)$.

[C. U. 1937.]

[The $\triangle pSp'$ contains a right angle at S and SX is the altitude.

$$\therefore \ pX.p'X = SX^2 = 4AS^2. \ ]$$

6. The intercept on the directrix of a parabola between the straight lines joining the ends of any given chord to a variable point on the parabola subtends a constant angle at the focus.

[Let PP' be any chord of a parabola, and Q any variable point on it. Join PQ, $P'Q$ meeting the directrix in p and p' respectively. Join PS, $P'S$ and produce them to meet the parabola in q and q'.

Join QS. Then pS, $p'S$ are bisectors of the angles QSq and QSq'.

Hence pp' subtends a constant angle at the focus, which is equal to half the angle subtended by PP' at the focus.]

7. State and prove the converse of the Proposition VIII.

8. Show that no straight line can meet a parabola in more than two points. [See *Fig. 11.*]

[If D be any point on the directrix and Q' any point on the parabola, then DQ' will meet the parabola in general in the second point Q which lies on qS produced, so that DS in the bisector of the angle $Q'Sq$. [*Ex. 7, Prop. VIII.*] When $Q'D$ is parallel to the axis, $Q'D$ and qS being parallel can but meet at infinity.]

9. Q and Q' are any two points on a parabola. If the straight line QQ' is divided at O in the ratio of the focal distances of Q and Q' respectively, shew that DS which is perpendicular to OS, the directrix and QQ' produced are concurrent.

[DS bisects the angle QSQ' externally.]

10. Given the focus and the directrix of a parabola, construct the parabola by Prop. VIII.

[Let DS bisect the $\angle ASq$. DA and qS meet on the parabola.]

TANGENCY.

Definition. *A tangent to a conic is a straight line which cuts the curve in two consecutive or coincident points.*

Note 1. Let AB be any secant to a conic, cutting the curve at P and Q. Now let AB move parallel to itself so that P and Q gradually approach each other. Ultimately in the position $A'B'$ the two points P, Q become two consecutive or coincident points and the secant AB is said to be the tangent $A'B'$ to the conic

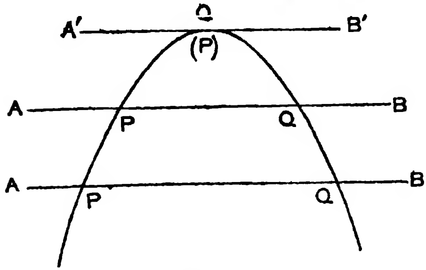

Fig. 11(a)

The point at which the tangent meets the curve is called the *point of contact*. The tangent $A'B'$ is said to touch the curve at the point of contact Q.

Note 2. The secant may be turned about a point A outside the curve.

When the secant is turned about the point A, the points P, Q at which the secant meets the curve gradually approach each other. Ultimately in the position AB' the secant becomes the tangent at the point Q.

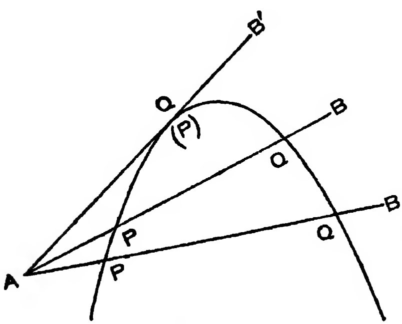

Fig. 11(b)

Note 3. The secant may be turned about one of its points of intersection with the curve.

As the secant AB is turned about the point P on the curve the two points P and Q gradually approach each other.

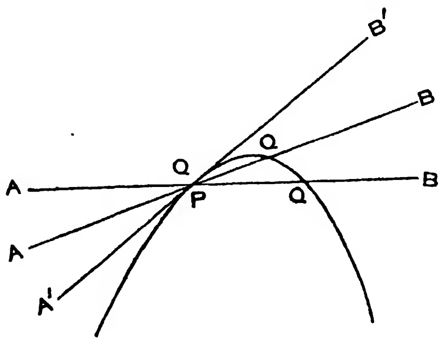

Fig. 11(c)

In the limiting position of the secant the points P and Q become consecutive or coincident; and the secant AB becomes the tangent $A'B'$ touching the curve at the point of contact P.

Proposition IX.

*The tangent to a parabola at any point is parallel to the system of chords bisected by the diameter through that point.**

Let PVP' be one of a system of parallel chords of a parabola, bisected by the diameter BV.

It is required to prove that the tangent at B is parallel to PVP'.

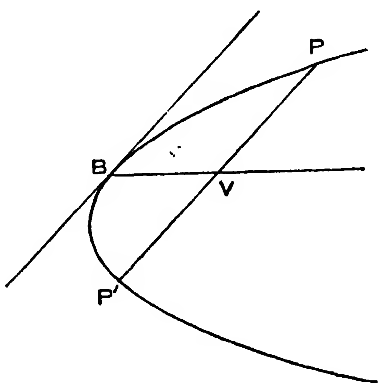

Fig. 12

Let the chord PVP' move parallel to itself, so that P' approaches B and ultimately coincides with it.

In each of its successive positions $PV = P'V$, because BV is the diameter of the system of chords parallel to PVP'.

Hence it is clear when P' coincides with V, P also coincides with V and the chord in its limiting position becomes the tangent to the parabola at B.

* C. U. 1928, '36.

$\therefore$ the tangent at B is parallel to the system of chords which are parallel to PVP'.

Cor. *There can be only one tangent to a parabola at a given point.*

If possible, let there be two tangents at any point B. Let BV be the diameter at B. Then the two tangents at B will be parallel to the same system of parallel chords, which is impossible. Hence there will be only one tangent at any point B.

EXERCISES.

1. *Draw a tangent to a parabola which will make a given angle with the axis.*

2. *Tangent at the vertex of a parabola is perpendicular to the axis.* [C. U. 1922, '36.]

3. If Q is a point on a parabola, shew that the tangent to the parabola at the vertex touches the circle described with diameter QS.

[From O the mid-pt. of QS, draw Om perpendicular to the tangent at the vertex and On perpendicular to the axis. Then Om = the radius of circle. If QN be the principal ordinate of Q, then $SQ = XN = 2AS + SN = 2AS + 2Sn = 2An = 2mO$ (Also $Am^2 = AN.AS$).]

4. *Draw a tangent to a parabola at a given point.*

5. The tangent at the vertex of a parabola is parallel to the directrix.

6. Shew that a tangent to a parabola can not be parallel to its axis. [*Prop. VIII, Ex. 8.*]

7. Draw a tangent to a parabola which will be parallel to a given straight line different from the axis. Show that only one such tangent is possible. [C. U. 1928.]

PROPOSITION X.

*The portion of the tangent to a parabola at any point intercepted between that point and the directrix subtends a right angle at the focus, and conversely.**

Let QZ be the tangent at Q meeting the directrix in Z.

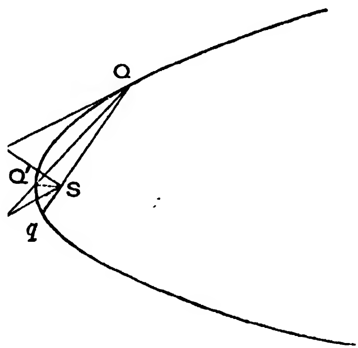

Fig. 13

Join ZS. Again join QS and produce it to meet the curve in q.

It is required to prove that the angle QSZ is a right angle.

Let QQ' be any chord of the parabola meeting the directrix in D.

Join DS and $Q'S$.

Now DS bisects the exterior angle $Q'Sq$, such that the $\angle Q'SD =$ the $\angle qSD$. [*Prop. VIII.*]

* C. U. 1932, '40.

Turn the secant $QQ'D$ about the point Q, so that Q' approaches Q.

In the limiting position when Q' coincides with Q the secant $QQ'D$ becomes the tangent QZ at Q and QD coincides with QZ.

Thus the $\angle$s $Q'SD$, qSD become the $\angle$s QSZ, ZSq respectively.

But the $\angle Q'SD$ is always equal to the $\angle qSD$.

$\therefore$ the $\angle QSZ =$ the $\angle ZSq$.

$\therefore$ the $\angle QSZ$ is a right angle.

THE CONVERSE TO PROP. X.

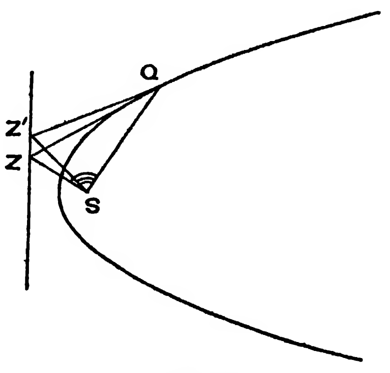

Fig. 14

Conversely, if a straight line QZ drawn from a point Q on a parabola to meet the directrix in Z subtends a right angle at the focus, the straight line is the tangent to the parabola at the point Q.

If not, let QZ' be the tangent at Q meeting the directrix at Z'.

Join $Z'S$. Then QZ' is the portion of the tangent lying between the curve and the directrix.

$\therefore$ the $\angle QSZ'$ is a right angle [*Prop. X*].

But by hypothesis, the $\angle QSZ$ is a right angle.

$\therefore$ the $\angle QSZ =$ the $\angle QSZ'$, which is absurd.

Hence QZ' can not be the tangent at Q.

$\therefore$ QZ is the tangent at Q.

EXERCISES.

1. (*a*) **Draw a tangent to a parabola at a given point.**

(*b*) **Draw a pair of tangents to a parabola from a point on the directrix. Show that the tangents are at right angles. [C. U. 1927]**

[(*a*) Draw SZ perpendicular to PS meeting the directrix in K.

(*b*) If Z be a point on the directrix, join ZS, and draw a focal chord perpendicular to it. Join ZP, Zp.]

2. Having given the tangent to a parabola with its points of contact and the directrix, show how to construct the curve.

3. ***Show that the tangents at the ends of the latus rectum meet the directrix at the point X at which the axis meet the directrix.***

[C. U. 1934.]

Proposition XI.

*The tangent at any point of a parabola bisects the angle between the focal distance of the point and the perpendicular drawn from the point on the directrix.**

Conversely, *the bisector of the angle between the focal distance of a point and the perpendicular drawn from the point on the directrix is the tangent to the parabola at that point.*

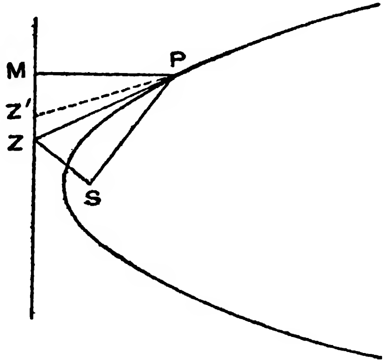

Fig. 15

Let the tangent at any point P on a parabola meet the directrix in Z.

Draw PM perpendicular to the directrix. Join PS.

It is required to prove that PZ bisects the angle MPS.

Join ZS.

Since PZ is the portion of the tangent intercepted between the point P and the directrix.

PZ subtends a right angle at the focus. [*Prop. X.*]

* C. U. 1911, '15, '22, '23, '26, '30, '33, '35, '38.

Now in the right-angled $\triangle$s PMZ and PSZ,

$PM = PS$,

PZ is common,

and the right angle PMZ = the right angle PSZ ;

$\therefore$ the $\triangle$s are equal in all respects.

$\therefore$ the $\angle MPZ$ = the $\angle SPZ$.

Hence PZ bisects the angle MPS.

Proof of the Converse Propositions.

Let PZ bisect the angle MPS.

It is required to prove that PZ is the tangent at P to the parabola.

In the $\triangle$s MPZ, SPZ, we have

$PM = PS$,

PZ common,

and the included $\angle MPZ$ = the included $\angle SPZ$.

$\therefore$ the $\triangle$s are equal in all respects.

Hence the $\angle PSZ$ = the $\angle PMZ$.

But the $\angle PMZ$ is a right angle.

$\therefore$ the $\angle PSZ$ is a right angle.

$\therefore$ PZ is the tangent at P to the parabola. [*Prop. X.*]

Cor. *The tangent at the vertex is at right angles to the axis.*

[C. U. 1915, '22.]

[The angle between the focal distance and the perpendicular distance of the vertex from the directrix is a straight angle. The tangent at the vertex bisects this angle.]

EXERCISES.

1. *Show that the tangent at any point P on a parabola bisects at right angles the straight line joining the focus to the foot of the perpendicular drawn from P on the directrix.* [C. U. 1935.]

Note. *Hence the locus of the reflection of the focus on any tangent is the directrix.* [C. U. 1923.]

2. Draw a tangent to a parabola at a given point on it.

[Draw the bisector of the $\angle MPS$.]

3. If PM be the perpendicular to the directrix from any point P on a parabola, shew that the locus of points equidistant from M and S is the tangent at P.

4. If PZ be a tangent to a parabola, meeting the directrix at Z, shew that Z is equidistant from the focus and the feet M and m of the perpendiculars drawn from the ends of the focal chord PSp on the directrix.

Hence construct tangents to a parabola from a point on the directrix.

[The triangles PMZ, PSZ are congruent.

$\therefore\ MZ = ZS$. Similarly, $ZS = Zm$; $\because$ pZ is also a tangent.

To construct the tangents from any point Z on the directrix, join ZS. Cut off ZM or $Zm = ZS$. Find P, p.]

5. (1) P and p are any two points on a parabola, the tangents at which meet at O. PM and pm are drawn perpendiculars on the directrix. Show that M, m and S are equidistant from O.

(2) Show how to construct two tangents to a parabola from a point outside it. [C. U. 1930.:]

Hints. (1) The Δs PMQ, PSO are congruent. $\therefore\ OM = OS$. Similarly $SO = Om$.

(2) To construct two tangents from an external point, draw a circle with centre O and radius OS, cutting the directrix at M and m. Then PO and Op are the tangents from O.

6. If the tangent to a parabola meets the directrix in Z, show that it bisects the angle between the focal distance of Z and the directrix.

7. If the tangent to a parabola at P meets the axis at T, shew that $SP = ST$. Hence prove that PT bisects SM at right angles, where M is the foot of the perpendicular PM on the directrix.

8. Draw a tangent to a parabola parallel to a given straight line.

If the given straight line makes an angle θ with the axis, draw SP making an angle 2θ with the axis and meeting the parabola at P. The tangent at P is the required tangent. [*See Ex. 7.*]

9. *If a leaf of a book be folded, so that one corner moves along on opposite side the line of crease touches a parabola of which the corner is the focus and the opposite side is the directrix.*

[Let $MXSK$ be a leaf of which the corner S moves on the opposite side MX. Let one of the positions occupied by S be m, the corresponding crease being QR.

Draw Pm perpendicular to MX meeting QR in P. Join PS.

The $\triangle$s QSR, QmR are equal in all respects, one being the exact reflection of the other.

$\therefore$ $mQ = QS$ and the $\angle mQR =$ the $\angle SQR$.

The $\triangle$s mQP, SQP are therefore equal in all respects.

$\therefore$ $mP = PS$.

It may be easily proved that the $\angle mPR =$ the $\angle SPR$...(1)

Now we see that P is a point such that mP is always equal to PS and the relation (1) also always holds good.

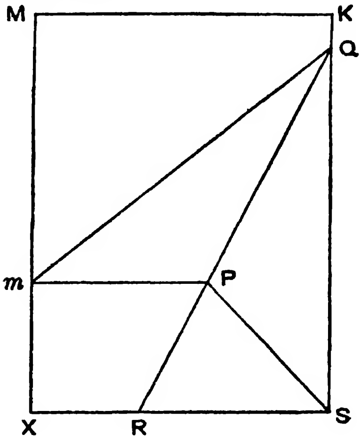

Fig. 16

$\therefore$ P is a point on a parabola of which S is the focus, MX the directrix, and the crease QPR is the tangent at P.

10. Find the locus of points equidistant from SP and MP.

Note. Two curves are said to be **confocal**, when they have the same focus.

Two curves are said to intersect **orthogonally**, when the tangents to the curves at the point of intersection are perpendicular to each other.

11. *Two* ***confocal parabolas*** *having the axes in the same straight line but in opposite directions, intersect orthogonally.*

[Let P and Q be the points of intersection of two confocal parabolas. Through P draw YPy parallel to the axis. Let PT_1 and PT_2 be the tangents to the two curves at P.

Then the $\angle YPT_1 =$ the $\angle T_1PS$, and the $\angle yPT_2 =$ the $\angle T_2PS$.

$\therefore$ the $\angle T_1PT_2$ is a right angle.]

12. *The tangent at any point of a parabola meets the directrix and the latus rectum produced at points which are equidistant from the focus.*

[Let the tangent at P meet the directrix at Z and the latus rectum produced at Y.

Draw PM perpendicular to the directrix.

The Δs PMZ, PSZ, are equal in all respects.

$\therefore$ the $\angle PZS =$ the $\angle PZM =$ the $\angle ZYS$.]

13. *If the ordinate PN of any point P be produced to meet the tangent at the end of the latus rectum at Q, show that $QN = SP$ (focal distance of P.)* [C. U. 1911.]

[The tangents at the ends of the latus rectum meet the axis at X. [*Prop.* X, *Ex.* 3.]

From similar Δs QNX, LXS

$$\frac{QN}{LS} = \frac{XN}{XS}. \quad \text{But } LS = XS \;;\; \therefore\; QN = XN = SP. \,]$$

14. *The locus of the foot of the perpendicular drawn from the focus on the tangent at any point is the tangent at the vertex.* [C. U. 1931.]

[Let SY be the perpendicular on the tangent at P and PM the perpendicular on the directrix.

The Δs MPY, PSY are equal in all respects.

$\therefore$ the $\angle PYM =$ the $\angle PYS$, whence each of them is a right angle.

$\therefore$ Y is the mid-pt. of MS, whence AY which is parallel to the directrix MX, is tangent at the vertex.]

15. *Having given the vertex of a diameter and a corresponding double ordinate, construct the parabola.*

[Let QVQ_1 be the double ordinate and BV the corresponding diameter. Through B draw a straight line YBY_1 parallel to QVQ_1, which is the tangent at B. Now apply Propositions XI, VII.]

[*Prop.* IX.]

16. *The tangent at the ends of a focal chord of a parabola intersect on the directrix at right angles.* [C. U. 1917, '19, '32, '33, '38.]

[Let PSp be any focal chord of parabola of which S is the focus and MZ the directrix.

Let PZ be the tangent at P meeting ths directrix in Z.

Join Zp.

Draw MP, mp perpendiculars on the directrix.

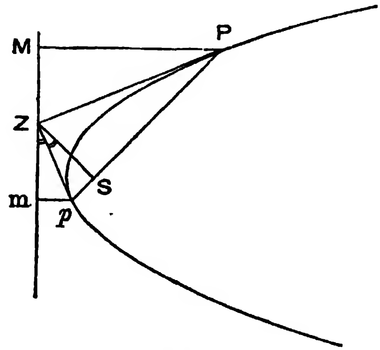

Fig. 17

It is required to prove that pZ is the tangent to the parabola at p and pZ is perpendicular to ZP.

Since PZ is the tangent at P,

$\therefore$ the $\angle PSZ$ is a right angle. [*Prof. X.*]

$\therefore$ the angle pSZ is also a right angle.

Thus pZ subtends a right angle at the focus S.

Hence pZ is the tangent at p. [*Prop. X.*]

To prove the second part of the proposition we proceed thus.

In the right-angled Δs MPZ and SPZ,

the $\angle MPZ =$ the $\angle ZPS$. [*Prop. XI.*]

$\therefore$ the $\angle MZP =$ the $\angle PZS$, each being complementary to equal angles.

Similarly the $\angle mZp =$ the $\angle pZS$.

$\therefore$ the $\angle MZP +$ the $\angle mZp =$ the $\angle PZS +$ the $\angle pZS$.

But $\angle mZM$ is a straight angle.

$\therefore$ the $\angle PZS +$ the $\angle pZS =$ one right angle.

Hence pZ is at right angles to PZ.

Note. (i) The locus of the point of intersection of the tangents at the ends of the focal chord of a parabola is the directrix.

(ii) The locus of the point of intersection of perpendicular tangents to a parabola is the directrix. [C. U. 1927.]

EXERCISES.

1. Show that the tangents at the extremities of a focal chord meet the directrix at the point where the diameter bisecting the focal chord meets it. Hence show that the tangents at the extremities of the latus rectum meet the axis on the directrix.

In fig. 17, Z is the pt. of intersection of the focal perpendicular on Pp with the directrix.

$\therefore$ Z is the point of intersection of the diameter with the directrix. [*Prop. V.*]

2. *Show that the circle described on any focal chord as diameter touches the directrix.* [C. U. 1917.]

In fig. 17, since the $\angle pZp$ is a right angle,
the circle with diameter Pp, passes through Z.

The $\angle pZm$ (= the $\angle pZS$) is equal to the angle ZPS in the alternate segment, each being complementary to the $\angle ZpS$.

3. *The portion of the tangent at any point of a parabola intercepted between the curve and the directrix is a mean proportional between the focal distance of the point and the focal chord through that point.*

In fig. 17, $ZP^2 = ZS^2 + SP^2 = SP.pS + SP^2 = SP.Pp$.

4. Shew that in the figure of Prop. XII, Sm and SM are respectively parallel to ZP and Zp.

5. The vertex of a diameter bisects the portion of the diameter intercepted between its parameter and the tangent at either extremity of the parameter.

6. The difference of the squares of the tangents at the extremities of a focal chord is equal to the rectangle contained by the focal chord and a parallel chord through the vertex.

If PSp be the focal chord of a parabola, the difference of the squares of the tangents at the extremities of the focal chord $= Pp.SP - Pp.Sp$ [*Ex. 3.*] $= Pp.(SP - Sp.)$:[*Apply Prop. V, Ex. 4.*]

7. Given a focal chord of a parabola and the tangents at its extremities, determine any number of points on it.

Find S drawing the focal perpendicular ZS. Find the directrix.

8. In the figure of Prop. XII, show that Mm is bisected at Z. [$MZ = ZS = Zm.$] [*Prop. XI, Ex. 5.*]

9. *Show that tangents at the extremities of any focal chord cut off equal intercepts on the latus rectum.* [*Apply Ex. 12, Prop. XI.*]

Each of the intercepts on the latus rectum $= ZS$.

The Subtangent.

The **subtangent** *at any point of a conic is the portion of the axis intercepted between the tangent and the ordinate of that point.*

Note. The subtangent (TN) is the projection of the tangent (PT) on the axis.

Proposition XII.

The subtangent at any point of a parabola is bisected at the vertex.* ($TA = AN$).

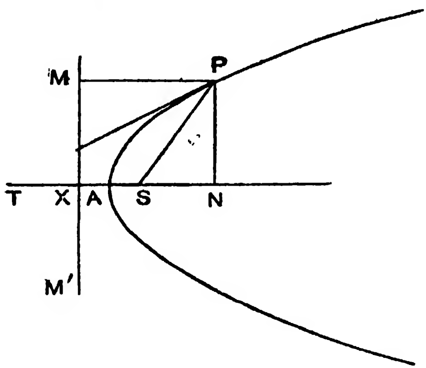

Fig. 18

Let the tangent at any point P of a parabola of which S is the focus A the vertex and MXM' the directrix meet the axis in T.

Draw PN, ordinate to P, meeting the axis in N. Then TN is the subtangent at P.

It is required to prove that the subtangent TN is bisected at A.

Join SP and draw PM perpendicular to the directrix meeting it in M.

*C. U. 1909, '18, '20, '25, '31, '37, '39.

Since PT is the tangent to the parabola at P, the $\angle MPT$ = the $\angle TPS$. [*Prop.* *XI.*] ... (i)

But MP is parallel to the axis TXN and PT meets them. $\therefore$ the $\angle MPT$ = the $\angle PTS$. ... ... (ii)

From (i) and (ii), the $\angle TPS$ = the $\angle PTS$.

$\therefore$ $PS = TS$. ... ... (iii)

Again, P being a point on the parabola,

$PS = PM = XN$. ... (iv)

From (iii) and (iv), $TS = XN$.

Now taking away the part XS common to TS and XN, we have $TX = SN$. ... ... (*a*)

Again A being the middle point of XS, $XA = AS$ ... (*b*)

From (*a*) and (*b*), by addition, $TA = AN$.

Hence the subtangent is bisected at the vertex.

Cor. ***The subtangent at any point of a parabola is double the abscissa of that point.*** **($TN = 2AN$.)**

EXERCISES.

1. (i) PT is the tangent at any point P, meeting the axis in T. PN the ordinate ; shew that $TX = SN$.

(ii) Draw a tangent at any point of a parabola with the help of this proposition. [C. U. 1937]

2. *From the middle point of AX, a tangent is drawn to a parabola ; shew that the length of the subtangent is equal to AS.*

If the tangent touches the parabola at P then the ordinate of P meets the axis at the middle point of AS.

3. Find the radius of the circumcircle of the ΔPTN, where PT is the tangent at any point P, meeting the axis in T and N is the foot of the ordinate from P.

[Through A, draw AR parallel to the ordinate PN, meeting the tangent in R. Then R is the circumcentre.] (Radius = $\frac{1}{2}PT$.)

Note. Since $PT^2 = PN^2 + TN^2 = 4AS.AN + 4AN^2 = 4AN.SP$,

$$\therefore \quad \frac{PT}{2} = \sqrt{SP.AN}.$$

4. *The tangent at any point P of a parabola meets the axis at T. Find the locus of the middle point of PT.* [C. U. 1925, 39.]

[*See Prop. XI., Cor.*]

Note. The focal perpendicular SY bisects PT. Hence the locus of the foot of the focal perpendicular upon the tangent PT is the tangent at the vertex. [*See Ex. 14, Miscellaneous Ex.*]

5. *Given the vertex, a tangent and its point of contact, construct the curve.* [C. U. 1909.]

[If P be the point of contact, join AP and produce PA to Q making PQ equal to $2AP$. On AQ as diameter describe a circle meeting the tangent at P in T. Then AT is the axis of the parabola. Apply Prop. XI to find the focus.]

6. *Find the locus of the point of intersection Q of the diameter through any point P with the perpendicular AY drawn from the vertex A on the tangent at the point P.*

[SM bisects PT at right angles. $\therefore$ SM is parallel to AQ, also PQ is parallel to AS.

$\therefore$ $MQ = AS$. $\therefore$ the locus of Q is a st. line parallel to the directrix on the side of it remote from A.]

7. The tangent at any point P meets the tangent at the vertex in O; show that $AO^2 = AS.AN$, where N is the foot of the ordinate of P. [$AO = \frac{1}{2}PN$] [C. U. 1918.]

8. If PM be the perpendicular from any point P on the directrix, shew that SM meets the tangent at the vertex on the tangent at P. Hence shew that the point of intersection is equidistant from T and N, the foot of the ordinate of P. [*Apply Prop. XI, Cor., Ex. 7.*]

9. Prove that the locus of Q, the point of intersection of a straight line through the focus parallel to the tangent at any point P and the diameter through P is a parabola.

[If Qn be perpendicular to the axis, the Δs PTN, QSn are equal in all respects, so that $Sn = TN = 2AN$.

Then $Qn^2 = PN^2 = 4AS.AN = 2AS.Sn = 4SS'.Sn$, where $2SS' = AS$.

Hence the locus of Q is a parabola having the same axis, the vertex being the point S and the latus rectum equal to half that of the original parabola.]

10. Draw a pair of tangents to a parabola from a point T on the axis. [Cut off $AN = AT$.] [C. U. 1921.]

Properties of Normals

Def.: *The* **normal** *to a conic at any point is the straight line drawn through the point perpendicular to the tangent at the point.*

PROPOSITION XIII

The normal to a parabola at any point makes equal angles with the focal distance of the point and the axis.

[C. U. 1917.]

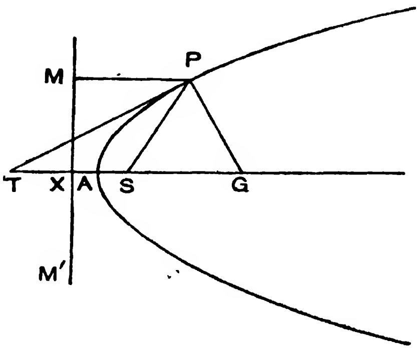

Fig. 19

Let the normal PG to the parabola of which PT is the tangent at P, S the focus, MXM' the directrix, meet the axis in G. Join PS.

It is required to prove that the angle SPG = the angle PGS.

Draw PM perpendicular to the directrix.

Since PM is parallel to the axis TS and PT meets them.

$\therefore$ the $\angle MPT$ = the $\angle PTS$ ··· ··· (1)

Again, since PT is tangent to the parabola at P,

the $\angle MPT$ = the $\angle TPS$ ··· ··· (2)

From (1) and (2), the $\angle TPS$ = the $\angle PTS$.

But in the $\triangle PTG$, the $\angle TPG$ being a right angle, the

$\angle TPS$ is complementary to the $\angle SPG$ and the $\angle PTG$ is complementary to the $\angle PGT$.

$$\therefore \text{ the } \angle SPG = \text{the } \angle PGS.$$

Hence the normal makes equal angles with the focal distance and the axis.

Cor. *The focal distance of any point of a parabola is equal to the portion of the axis intercepted between the focus and the normal at the point.* ($SP = SG$.)

Note. The phrase 'normal at a point' is usually used for 'the portion of the normal intercepted between the curve and the axis.'

EXERCISES.

1. Shew that the normal at any point of parabola is twice the distance of the tangent from the focus.

[Draw perpendicular from the focus on the tangent and the normal at P.]

2. If PM be drawn perpendicular on the directrix from any point P on a parabola, shew that the normal at P is parallel and equal to SM. Hence show that MG and SP bisect each other.

[If PG be the normal at P, SG is equal and parallel to MP, so that $MPGS$ is a parallelogram.]

3. Show that the normal at any point of a parabola bisects the angle between the diameter through that point and the focal distance of it.

[The angle which the diameter makes with $PG =$ the $\angle PGS$.]

4. Shew that the feet of the perpendiculars drawn from the focus on the tangent and the normal at any point, lie on a diameter.

[The perpendiculars meet PG and PT at their mid-pts.]

5. Shew that if $PG = PM$ (= the focal distance of P), MG bisects the focal distance at right angles. Hence shew that T, M, P and G are concyclic.

[The $\triangle PSG$ is equilateral and each of $MPGS$ and $MPST$ is a rhombus.]

6. Shew that the portion of any straight line drawn through S and intercepted between the tangent at any point P and a straight line Gt drawn through G parallel to the tangent is bisected at the focus S.

Through S draw a straight line ll' meeting PT, Gt in l and l'. The Δs TSl, GSl' are equal in all respects, so that $Sl = Sl'$.

7. *A chord PQ of a parabola is normal to the curve at P and subtends a right angle at the focus S; show that $SQ = 2SP$.* [C. U. 1917]

In the figure of Proposition XIV, produce PG to meet the curve in Q Join SQ. Draw QR perpendicular on MP produced to meet it in R. Draw QN perpendicular on the directrix. The $\angle RPG =$ the $\angle SGP =$ the $\angle SPG$ (Prop. XIV, Ex. 3) ; and SPQ, PRQ are right-angled Δs, PQ being common to both. $\therefore$ the Δs are congruent.

$\therefore$ $PR = SP$. $\therefore$ $SQ = NQ = MR = MP + PR = SP + SP = 2SP$.

8. Show that the foot of the perpendicular drawn from G, the point of intersection of the normal at one end of a focal chord with the axis, on the tangent at the other end of the focal chord lies on the latus rectum.

The tangents PZ, pZ at the ends of the focal chord PSp meet the latus rectum at l and l' respectively. Join $l'G$. The Δs ZSP, GSl' are congruent. [*Prop. XI, Ex. 12.*]

9. Show that PG is tangent to the circle $PSZM$.

[The $\angle SPG =$ the $\angle PSZ$, each being complementary to the angle ZPS.]

10. Show that the straight line drawn through G parallel to the tangent at P touches an equal confocal parabola.

[Let the straight line GQ parallel to the tangent PT at P meet PS produced in Q. Then Q is a point on the confocal equal parabola corresponding to P. TQ is parallel to PG and is normal at Q, while QG is the tangent at Q.

11. Show that $ST = SP = SG$. Hence prove that SM and PT bisect each other at right angles.

12. *The normals at the ends of a focal chord meet on the diameter bisecting that chord.*

[The diameter becomes coincident with one of the diagonals of the rectangle formed by tangents and normals.] [*Prop. XII.*]

13. If Pf and Pf' be drawn to the axis, making equal angles with the normal PG, prove that $SG^2 = Sf.Sf'$.

[The $\triangle$s SPf, SPf' are similar.]

14. *Shew how to draw the normal at any given point without drawing the tangent.* [C. U. 1916.]

If P be the given point on the parabola, with centre S and radius SP describe an arc of a circle cutting the axis at G. Then PG is the required normal at P.

PROPERTIES OF SUBNORMALS.

Definition : *The* **Subnormal** *at any point of a conic is the portion of the axis intercepted between the normal and the ordinate of that point.*

Note. The subnormal (NG) is the projection of the portion (PG) of the normal between the point P and its point of intersection with the axis, on the axis of the curve.

PROPOSITION XIV.

**The subnormal at any point of a parabola is equal to the semi-latus rectum.* ($NG = 2AS$.)

Let P be any point of a parabola of which S is the focus and MX the directrix. Draw PG normal at P meeting the axis at G and PN the ordinate. Then NG is the subnormal.

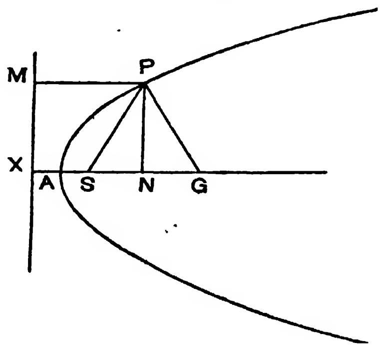

Fig. 20

It is required to prove that $NG = 2AS$. *Join PS and draw PM perpendicular to the directrix.*

Since PG is the normal at P,

$\therefore$ the $\angle SPG =$ the $\angle SGP$. [*Prop. XIII*]

$$\therefore \ SP = SG.$$

$$\text{But } SP = PM = XN.$$

$$\therefore \ XN = SG,$$

$$\text{or, } XS + SN = SN + NG.$$

$$\therefore \ NG = XS = 2AS.$$

* C. U. 1916, '19, '20, '24, '26, '29.

EXERCISES.

1. *Show how to draw the normal at any point of a parabola without drawing the tangent, the axis of the parabola being given.* [C. U. 1916.]

2. *Draw a normal to a parabola making a given angle with the axis.*

[In the ΔSPG, the $\angle PSG = 180^\circ - 2\angle SGP$.]

3. Show that the projection of the normal on the focal distance $= 2AS$.

4. *Show that the locus of the foot of the perpendicular from the focus on the normal is a parabola, whose vertex is S and latus rectum $= AS$.*

[Let Sp be the perpendicular from S on PG and pn the perpendicular from p on SG. $\therefore$ $pn^2 = \frac{1}{4}PN^2 = AS.AN = AS.Sn$. Hence, etc. [*Prop. IV, Ex. 1.*]

5. If from one extremity P' of the double ordinate PNP' of a parabola, a perpendicular is drawn on the tangent at the other extremity to meet the diameter through P at K, show that the portion KP of the diameter through P intercepted between the point P and the perpendicular is a constant.

[The Δs PNG, $PP'K$ are similar.]

6. From any point P on a parabola, PM is drawn perpendicular to the directrix. A portion PO equal to $2AS$ is cut off from PM; show that XO is parallel to PS and ON is parallel to the bisector of the angle PSX.

[$OP = NG = XS$.]

7. In Exercise 5, find the locus of K.

[The locus is an equal parabola having A' for its vertex, where $AA' = 4AS$.]

8. *Prove that the normal to a parabola at any point is mean proportional between the focal distance of the point and the latus rectum.*

[Let PN be the ordinate of any point P of a parabola.

Now $PG^2 = PN^2 + NG^2 = 4AS.AN + 4AS^2 = 4AS.XN = 4AS.SP$.]

9. Show that, if the normal PG be equal to the focal distance of P, SG is equal to the latus rectum and conversely.

10. *If the triangle SPG be equilateral, then SP is equal to the latus rectum.* [C. U. 1920.]

[$SN=NG=2AS$.]

11. *If the ordinate of a point Q bisects the subnormal of a point P, the ordinate of Q is equal to the normal at P.* [C. U. 1919.]

[Let Qn, the ordinate of Q, bisect the subnormal NG at n.

Then $Qn^2=4AS.An=4AS.XN$ ($\because$ $Nn=AS$)

$=4AS.SP=PG^2$. [*Ex.* 8.]

12. The perpendicular from P to the chord AP meets the axis at K. Show that $GK=NG=2AS$.

[The $\triangle$s APN, PNK are similar.]

13. If Sn be drawn perpendicular to the normal PG at any point P, shew that $Sn^2=SP.AN$.

[$Sn=\frac{1}{2}PT$. But $PT^2=4AN.SP$, [*Prop.* XII, *Ex.* 3.]

$\therefore$ $SN^2=AN.SP$.]

CHAPTER II

THE ELLIPSE.

Definition : *An* **Ellipse** *is a conic whose eccentricity is less than unity.*

PROPOSITION I.

*Given the focus, the directrix and the eccentricity of an ellipse, to determine any number of points on the curve from the definition.**

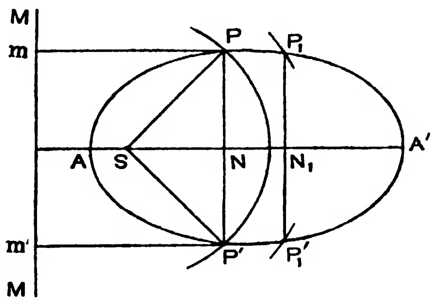

Fig. 21

Let S be the focus, MXM' the directrix and e the eccentricity of an ellipse.

It is required to find any number of points on the curve.

From S draw a straight line SX perpendicular to the directrix, meeting it in X.

* C. U. 1913.

Divide SX in the ratio of $e:1$. But SX may be divided in the given ratio in two ways *viz* :—

(i) internally at A, so that $\frac{SA}{AX}=e$, and

(ii) externally at A', so that $\frac{SA'}{A'X}=e$.

Since A and A' are two points satisfying the given condition, they are points on the ellipse.

Now in AA' take any point N and through N draw PNP' perpendicular to AA'.

With centre S and radius $e.XN$ describe an arc of a circle cutting PNP' at P and P'.

Then P and P' are points on the ellipse.

Join PS and $P'S$.

From P and P', draw Pm and $P'm'$ perpendiculars to the directrix.

Now $SP=e.XN=e.Pm$; $SP'=eXN=eP'm'$,

$$\therefore \quad \frac{SP}{Pm}=e\ ; \quad \text{and} \quad \frac{SP'}{P'm'}=e.$$

In other words, the focal distances of P and P' bear to their respective distances from the directrix a constant ratio which is equal to the eccentricity e.

Hence P and P' are points on the ellipse.

Proceeding exactly in a similar way, by drawing through any other point N_1 in AA' a straight line perpendicular to it, we can find two other points on the curve. In like manner, any number of points on the curve may be obtained.

Note 1. The straight line AA' is called the **major axis** of the ellipse. The points A and A' are called the **vertices** of the ellipse, these being the points of intersection of the major axis with the curve.

Hence the **major axis** of an ellipse is the portion of the axis intercepted between the vertices (A and A').

Note 2. The eccentricity e is less than unity, so that $SA < XA$ and $SA' < XA'$. In other words, the distances of A and A' from the focus are less than their distances from the directrix.

That is, A and A' are nearer to the focus than to the directrix. Therefore A and A' must lie on the same side of the directrix as the focus.

EXERCISES.

1. *Prove that the axis of an ellipse cuts it in two points only.*

If possible, let the axis cut the ellipse at a third point A_1 different from A, A'. Two cases may arise according as A_1 is supposed to lie between S and A', or between S and A.

CASE (A). When A_1 is between S and A',

$$SA' = eA'X \text{........................ (I)}$$

$$SA_1 = e.A_1X \text{..................... (II)}$$

Subtracting (II) from (I), we get

$$SA' - SA_1 = e(A'X - A_1X), \quad \text{or,} \quad A_1A' = eA_1A',$$

or, $A_1A'(1-e)=0$, which is absurd, for neither $(1-e)$ nor A_1A' can be equal to zero.

[$\because$ $e < 1$ and A_1 is different from A and A'.]

CASE (B). When A_1 is between S and A,

$$SA_1 = e.A_1X \text{.................. (I)}$$

$$SA = e.AX \text{................. (II)}$$

Subtracting (I) from (II), we have

$$SA - SA_1 = e.(AX - A_1X), \text{ or, } AA_1 = -e.AA_1.$$

$$\therefore \quad AA_1(1+e) = 0.$$

But $\quad (1+e)$ can not be zero ;

$\therefore \quad AA_1 = 0$, which is absurd, for A_1 is different from A.

Hence the axis cuts the ellipse in two points only.

2. Construct the ellipse having given the following data :

(i) The eccentricity, the directrix and two points on the curve. (Two ellipses satisfy the condition).

(ii) The eccentricity, the focus, and the point where the axis meets the directrix.

(iii) The focus, the eccentricity and two points on the curve. [Find the directrix.]

(iv) The eccentricity, the focus and a vertex.

(v) The eccentricity, a vertex and the point X, where the axis meets the directrix. [Find the focus.]

(vi) The eccentricity, a vertex and the directrix. [*Prop. I.*]

Definitions :

Any finite straight line joining two points on an ellipse is called a **chord** *of the ellipse.*

Any chord of an ellipse passing through one of its foci is called a* **focal chord** *of the ellipse.*

3. A focal chord QSq of an ellipse meets the directrix at R ; show that $Sq.\ QR = QS.\ Rq$.

[Draw QM and qm perpendiculars to the directrix.

The $\triangle$s QMR, qmR are similar.]

*It will be seen later that an ellipse has two foci.

4. Prove that an ellipse lies only on one side of the directrix.
[*See Note 2, Prop. I.*]

5. *Shew that as a point moves on the ellipse from A to A', its focal distance increases from SA to SA'.*

[The focal distance, $SP = e.NX$. The value of NX increases from AX to $A'X$.]

6. *A point is outside, within or on the ellipse according as the ratio of its distance from the focus to its distance from the directrix is greater, less or equal to the eccentricity of the ellipse.* [C. U.]

[Follow the same method as in Exs. 11 and 12, Prop. I. Parabola.]

7. *If a parabola and an ellipse have the same focus and directrix, show that the parabola lies entirely outside the ellipse.*

Take any point P on the parabola ; join PS and produce it if necessary to cut the ellipse in P_1.

$\therefore$ P being a pt. on the parabola, $\dfrac{SP}{PM} = 1$ (eccentricity). P_1 being a pt. on the ellipse, $\dfrac{SP_1}{P_1M_1} = e$, which < 1.

$\therefore$ P must be outside the ellipse. [*Ex. 6.*]

8. If P be any point on an ellipse and PM perpendicular to the directrix, then the intercept made on SM or SM produced, by perpendiculars to AA' at A and A' subtends a right angle at P.

[Let Aa and $A'a'$ be perpendiculars to AA' at A and A' meeting SM or MS produced at a, and a' respectively. Join Pa, Pa'.

Thus Pa is the internal bisector of the angle MPS and Pa' is the external bisector of the angle MPS. Hence the result.]

Definition : *The* **Latus rectum** *of an ellipse is the focal chord perpendicular to the major axis of the ellipse.*

9. Show that the length of the latus rectum is $2(1+e)AS$.

[If LSL' be the latus rectum, draw LM perpendicular to the directrix. Thus,

$$LL'=2SL=2eML=2e(XA+AS)=2(AS+eAS)=2AS(1+e).\ ]$$

10. *Prove that the ellipse lies entirely between the two straight lines drawn perpendicular to the axis at A and A'. Hence prove that the ellipse is a closed curve.*

[*See the figure of Proposition I.*]

In order that the construction of the ellipse may be possible, SN must be less than $e.XN$ or SP (the radius of the circle with centre S).

If n_1 be the position of N to the left of A and n_2, the position of N to the right of A'.

$$\begin{aligned} Sn_1 &= SA+An_1=e.AX+An_1 \quad (\because\ e.AX=SA.) \\ &= e.(Xn_1+n_1A)+An_1 \quad (\because\ AX=An_1+n_1X) \\ &= e.Xn_1+(1+e)An_1. \end{aligned}$$

$\therefore$ we find $Sn_1 > e.\ Xn_1$, because $(1+e)\ An_1$ is positive.

$$\begin{aligned} \text{Again, } Sn_2 &= SA'+A'n_2=e.A'X+A'n_2 \quad (\because\ SA'=eA'X) \\ &= e.(Xn_2-A'n_2)+A'n_2=e.Xn_2+(1-e)\ A'n_2. \end{aligned}$$

$\therefore$ $Sn_2 > e.Xn_2$, since $(1-e)A'n_2$ is evidently positive, e being less than unity. Hence N can neither be taken to the left of A nor to the right of A', so that the points on the curve are exclusively between A and A'.

Again SP, the distance of any point on the curve from the focus $=e.XN$. As the point P on the curve moves from A to A', $e.XN$ increases from $e.AX$ to $e.A'X$; SP or $e.XN$ is always finite. Hence the ellipse is a closed curve. [*Ex.* 5.]

11. *The ellipse is symmetrical with respect to its axis major.**

[C. U. 1911.]

[*c.f. Prop. II, Chapter I.*]

12. *Any two right lines drawn from any point on the axis major to the curve, on opposite sides of the axis and equally inclined to it, are equal in length, and conversely.* [C. U. 1911.]

[*c.f. Ex.* 2, *Prop. II, Chapter I.*]

Definition : *The* **Centre** *of an ellipse is the middle point of the major axis. It is usually denoted by the letter C.*

The **Minor axis** *of an ellipse is the chord through the centre, perpendicular to the major axis.*

Note. The minor axis may be said to be the *central double ordinate* of an ellipse and is usually denoted by BCB'.

*The definition of 'symmetry' has already been given in Chapter I.

Proposition II.

An ellipse is symmetrical about its minor axis.

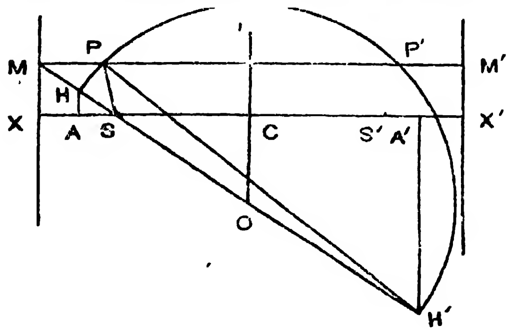

Fig. 22

Let MX be the directrix and S the focus of an ellipse and let A and A' be its vertices.

Through A and A' draw two perpendiculars to the axis SAX to cut any line MS at H and H' respectively, M being a point on the directrix.

Draw a circle on HH' as diameter. At M draw a st. line perpendicular to the directrix to meet the circle at P and P'. Then P and P' are both points on the ellipse.

Proof :—

$$\frac{HS}{HM}=\frac{AS}{AX}=e\ ;\ \frac{H'S}{H'M}=\frac{A'S}{A'X}=e, \text{ where } e=\text{eccentricity.}$$

$\therefore$ H, H' divide MS internally and externally in the same ratio.

Hence, from a well known proposition in geometry, we have $\dfrac{PS}{PM}=\dfrac{HS}{HM}$, because P is any point on the circle with HH' as diameter.

That is $\frac{PS}{PM}=e$, hence P is a point on the ellipse.

Similarly P' is also a point on the same ellipse.

Now, if O be the middle point of HH' and C be the middle point of AA', then OC is parallel to $A'H'$ and hence is perpendicular to AA'. But MPP' is parallel to AA'.

Therefore OC is perpendicular to the chord PP' of the circle with O as centre.

Hence OC bisects PP' perpendicularly. But OC being perpendicular to AA' at its middle point coincides with the minor axis of the ellipse. Again, P and P' being two points on the ellipse, PP' is also a chord of the ellipse and it is bisected perpendicularly by the minor axis.

Hence the ellipse is symmetrical with regard to the minor axis.

EXERCISES.

1. *Any two right lines drawn from any point on the minor axis to the curve equally inclined to and on opposite sides of the axis are equal in length and conversely.*

It follows from the symmetry of the ellipse about the minor axis.

[*Prop. I, Ex. 12.*]

2. *Every central chord of an ellipse is bisected at the centre.*

[Let PCp be any central chord. Draw a straight line CQ making the $\angle BCP=$ the $\angle BCQ$ and meeting the curve in Q. Now CQ and CP being equally inclined to the minor axis are equal in length.]

[*Prop. II, Ex. 1.*]

Note. A curve is said to be *symmetrical about a point* if the point bisects every chord passing through it. *An ellipse is symmetrical about its centre.*

3. Show that the common chords of an ellipse and a concentric circle which will in general cut in four points are either parallel to the axes or pass through the centre.

[This follows from the symmetry of the ellipse about its axes.

[*See Ex. 12, Prop. I, and Ex. 1, Prop. II.*]

4. Draw the major and minor axes of an ellipse, given

(1) the ellipse and a focus,

and (2) the ellipse and its centre.

(1) With the focus as centre and with any radius describe an arc of a circle cutting the ellipse in two points P_1 and P_2. Similarly describe another arc with a different radius meeting the ellipse in p_1 and p_2.

[Join the middle points of P_1P_2 and p_1p_2 to get the major axis.]

(2) With the centre of the ellipse as centre and with any suitable radius describe a circle cutting the ellipse in four points forming a rectangle.

Join the middle points of each pair of opposite sides.

[*Otherwise* : Follow the method given in (I)]

5. Straight lines joining the ends of any two central chords of an ellipse form a parallelogram.

Any two central chords of an ellipse, like the diagonals of a parallelogram, bisect each other. [*Prop. II, Ex. 2.*]

Proposition III.

*The ellipse has a second focus and a second directrix.**

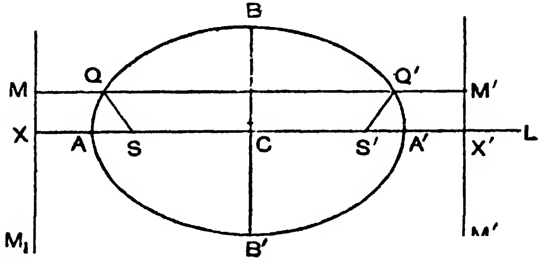

Fig. 23

Let S be the focus of an ellipse of which MXM_1 is the directrix.

It is required to prove that the ellipse has a second focus and a second directrix.

From S draw SX perpendicular to the directrix and produce XS to any point L, cutting the ellipse in A and A'.

Let C be the middle point of AA' and BCB' the minor axis. From XS produced cut off CS' equal to CS and CX' equal to CX. Through X' draw a straight line M'_1X' perpendicular to CX'.

Then S' is the second focus, and M_1X' the second directrix.

Let Q be any point on the ellipse. Through Q draw a straight line parallel to the axis meeting M_1X, M'_1X' in M and M' respectively and the ellipse in another point Q'.

Now fold the figure about the minor axis BCB'. Then S will fall on S', since $CS = CS'$, and $\angle$s BCS, BCS' are right angles. Similarly X will fall on X'.

* C. U. 1928.

Again, since the $\angle$s MXC and $M'X'C$ are right angles, MXM_1 coincides with $M'X'M'_1$.

By symmetry about the minor axis, Q falls on the point Q' of the ellipse ; M falls on M' also.

Hence $SQ=S'Q'$ and $MQ=M'Q'$. But QM and $Q'M'$ are the perpendicular distances of Q, Q' from MX and M'_1X' respectively.

$$\therefore \quad \frac{SQ}{QM}=\frac{S'Q'}{Q'M'}=\text{the eccentricity of the ellipse.}$$

$\therefore$ by definitions, M'_1X' is the second directrix and S' is the second focus of the ellipse.

EXERCISES.

1. If e be the eccentricity of the ellipse, show that $S'P'=ePM$, and $SP=S'P'$, where P and P' are two points symmetrically situated about the minor axis.

2. Shew that the quadrilateral formed by joining the two foci and the extremities of the minor axis is a rhombus.

PROPOSITION IV.

In an ellipse,

(*i*) $CA = e.CX$; (*ii*) $CS = e.CA$; (*iii*) $CS.CX = CA^2$.*

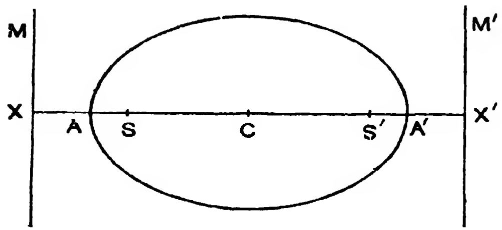

Fig. 24

Let C be the centre of an ellipse of which S and S' are the foci.

Since A and A' are points on the ellipse, from the definition, we have

$$SA = e.AX \quad \ldots \quad \ldots \quad (1)$$

$$\text{and} \quad SA' = e.A'X = e.AX'. \quad \ldots \quad (2)$$

(*i*) By adding (1) and (2), we have

$$SA + SA' = e.(AX + AX'),$$

$$\text{or,} \quad AA' = e.XX'.$$

$$\text{But} \quad AA' = 2CA \text{ and } XX' = 2CX.$$

$$\therefore \quad CA = e.CX. \quad \ldots \quad \ldots \quad (3)$$

(*ii*) By subtracting (1) from (2), we have

$$SA' - SA = e.(AX' - AX),$$

$$\text{or,} \quad SS' + S'A' - SA = e.(AA' + A'X' - AX)$$

*C. U. 1932.

or, $SS' = e.AA'$. [$\because$ $S'A' = SA$, and $A'X' = AX$.]

But $SS' = 2CS$ and $AA' = 2CA$.

$\therefore$ $CS = e.CA$. ... ... (4)

(*iii*) From (3) and (4), we have

$$CA = e.CX \text{ and } e.CA = CS.$$

By multiplication, we get $e.CA.CA = e.CX.CS$,

or, $CX.CS = CA^2$.

EXERCISES.

1. *Show that* $CB^2 = CS.CX.(1-e^2)$ *and* $e^2.CX = CS$.

2. *Show that the semi-major axis is equal to the line joining the focus to the extremity of the minor axis.* [$SB = CA$.] [C. U. 1932.]

3. Given the ellipse, eccentricity e and its centre C, find the foci. [Find the axes.]

4. *Given the ellipse and one focus, find the centre and the eccentricity.* [*Prop. IV, Ex. 4.*]

5. Given the ellipse, a directrix and the eccentricity, find the centre and foci.

[Draw two chords of the ellipse parallel to the directrix. The middle points of the parallel chords give the axis.]

Proposition V.

*The sum of the focal distances of any point on an ellipse is a constant and is equal to the major axis.**

$$(SP + S'P = AA')$$

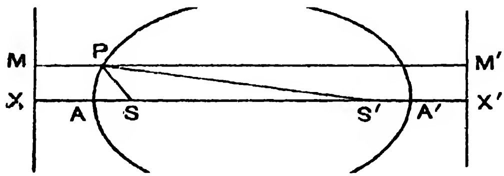

Fig. 25

Let S and S' be the foci of an ellipse of which MX and $M'X'$ are the directrices.

Let P be any point on the ellipse. Join SP and $S'P$.

It is required to prove that $SP + S'P = AA'$.

Through P, draw a straight line MM' perpendicular to the directrices, meeting them in M and M'.

From the property of the ellipse,

$$SP = e.PM, \text{ and } S'P = e.PM'.$$

By addition, $SP + S'P = e.(PM + PM')$

$$= e.MM'$$

$$= e.XX'$$

$$= 2e.CX$$

$$= 2CA \qquad [\textit{Prop. IV}]$$

$$= AA'.$$

Cor. $BS + BS' = AA'$.

*C. U. 1909, '10, '12, '15, '17, '20, '22, '24, '27, '30, '34, '35, '37.

EXERCISES.

1. To construct an ellipse mechanically. [C. U. 1910.]

The sum of the distances of any point P on an ellipse from the foci S and S' is always constant.

Hence an ellipse is a curve traced out by a point which moves in a plane, in such a way that the sum of its distances from two fixed points in the plane is constant.

Therefore an ellipse may be mechanically constructed as follows :

Take two pins and fasten the ends of a string to them. Fix the two pins at S and S', on a paper pinned on a drawing board.

Place a pencil point P against the string. Now trace a curve by moving the pencil point around, in such a way that it is always pressed against the string, so as to keep it stretched.

The curve will evidently be an ellipse with foci at S and S', and major axis equal to the length of the string.

$\because$ $SP+S'P=$ length of the string, which is always a constant,
$=AA'$, the major axis.

2. If a point moves in a plane in such a way that the sum of its distances from two fixed points in the plane is constant, then the locus of the point is an ellipse.

3. Prove that the major axis is the longest chord in an ellipse. [C. U. 1917, '20, '34.]

Let PQ be any chord of an ellipse having for its foci S and S'.

Then $PS+QS > PQ$, and $PS'+QS' > PQ$.

$\therefore$ $(PS+QS)+(PS'+QS') > 2PQ.$

$\therefore$ $(PS+PS')+(QS+QS')=AA'+AA'=2AA' > 2PQ.$

Hence AA' is greater than any other chord PQ.

4. The distance of either extremity of the minor axis from either focus is equal to half the major axis.

[C. U. 1909, '15, '32.] [*Prop. V, Cor.*]

5. Given the base and the sum of the other two sides of a triangle, find the locus of the vertex.

[The locus of the vertex is an ellipse, the extremities of the base being the foci.]

6. *A circle is drawn wholly within another circle. Find the locus of a point equidistant from the two circumferences.*

The point equidistant from the two circumferences is the centre of a circle touching the two circles.

Let S and S' be the centres and a, b the radii of the outer and the inner circles and P the centre and r the radius of any circle touching them at A and B respectively.

Now $$SP = SA - AP = a - r \quad \ldots \quad \ldots \quad \text{(i)}$$
$$S'P = S'B + BP = b + r \quad \ldots \quad \ldots \quad \text{(ii)}$$

From (i) and (ii), $SP + S'P = a + b =$ a constant.

7. *Shew that two ellipses having the same foci can not intersect each other.* [C. U. 1922.]

If P be any pt. of intersection of the two ellipses having the same foci S and S', $PS + PS' =$ the major axis.

$\therefore$ the major axes of the ellipses are equal, which is contrary to the hypothesis.

8. If P be any point on an ellipse, find when the angle SPS' is the greatest.

[The base and the sum of the sides of a triangle being given, the isosceles $\triangle$ has the greatest vertical angle.]

9. A point lies within, without or on an ellipse according as the sum of its distances from the foci is less than, greater than, or equal to the major axis. [C. U. 1912, '27, '30.]

Case I. If P is inside the ellipse, let SP produced intersect the ellipse at Q'. Join $S'Q$.

Then $(S'Q + PQ) > S'P$, $\therefore$ $(SP + S'P) < SP + (PQ + S'Q)$;

or, $(SP + S'P) < (SQ + S'Q)$. $\qquad (SP + S'P) < AA'$.

$$(\because SQ + S'Q = AA').$$

Case II. When P is outside the ellipse.

Join SP cutting the ellipse in Q. Join $S'Q$.

But $(SP+S'P)=(SQ+QP)+S'P.$

$\therefore$ $(SP+S'P)>(SQ+S'Q)$; ($\because$ $QP+S'P>QS'$)

$\therefore$ $(SP+S'P)>AA'$. ($\therefore$ $SQ+S'Q=AA'$).

Case III. When P is on the ellipse, $SP+S'P=AA'$.

10. *S and S' are two fixed points. With radius $2a$ (greater than SS') and centre S describe a circle. Take any radius SQ of the circle. From QS cut off QP equal to PS'. Find the locus of P.*

[$PS+PS'=SQ=$ a constant $=2a$. The .locus of P is an ellipse of which S and S' are foci.]

11. *If of two ellipses one falls entirely within the other, the axis major of the one is less than the axis major of the other.* [C. U. 1910.]

Hints. Let the major axis of the outer and the inner ellipses be AA' and DD' respectively. Produce DD' to meet the outer ellipse at d and d'. Join Sd, Sd', $S'D$, $S'D'$, where S, S' are the foci of the outer ellipse. $Sd+Sd'>dd'$; $S'd'+S'd>dd'$, and so on.

$\therefore$ $2AA'>2dd'$, whence $AA'>DD'$.

12. *If two ellipses have a common focus, and their major axes equal, show that they cannot intersect in more than two points.*

Let S be the common focus, S_1 and S_2 the second foci. If P be any point common to the two ellipses,

$$PS+PS_1=PS+PS_2 \text{ (= the major axis).} \quad \therefore \quad PS_1=PS_2.$$

Hence the common points between the two ellipses lie on the perpendicular bisector of S_1S_2, the line joining the second foci. If there be three points of intersection, they must be on the ellipses as well as on the perpendicular bisector of S_1S_2, which is absurd.

13. If the two foci of an ellipse coincide, show that the ellipse becomes a circle. [C. U. 1924, '34.]

Proposition VI.

*The semi-minor axis is mean proportional between the segments of the major axis by the focus.**

$(CB^2 = SA.SA'.)$

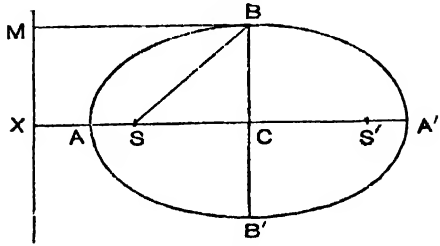

Fig. 26

Let C be the centre and S, S' the foci of an ellipse of which MX is the directrix and BCB' the minor axis.

Join BS. Draw BM perpendicular to the directrix.

It is required to prove that $CB^2 = SA.SA'$.

Since B is a point on the ellipse,

$$SB = e.BM = e.CX = CA. \qquad [\textit{Prop. IV}]$$

$$\therefore \; CB^2 = SB^2 - CS^2 = CA^2 - CS^2$$
$$= (CA + CS)(CA - CS)$$
$$= (CA' + CS)(CA - CS) \quad (\because \; CA' = CA)$$
$$= SA'.SA.$$

$$\therefore \; CB^2 = CA^2 - CS^2 = SA'.SA.$$

EXERCISES.

1. *Show that* $CB^2 = CS.XS$.

$$CB^2 = SB^2 - CS^2 = CA^2 - CS^2 = CA.CA - CS.CS$$
$$= CA.e.CX - CS.eCA \qquad [\textit{Prop. IV.}]$$
$$= e.CA(CX - CS) = e.CA.XS = CS.XS.$$

*C. U. 1916, '21, '26, '36.

2. Given the major axis of an ellipse in magnitude and position and the minor axis, find the foci and the directrices. [$BS=CA$.]

3. Show that $AA'^2=BB'^2+SS'^2$. [$CA^2=CB^2+CS^2$.]

4. *If circles are described on AA' and SS' as diameters, prove that the portion of the tangent to the inner circle lying within the outer circle $=BB'$.*

(Half the tangent)^2 = (radius of the outer circle)^2 − (radius of the inner circle)$^2 = CA^2-CS^2$.

5. *Find the length of the semi-latus rectum in terms of the axes.*
[C. U. 1926.]

From definition, $SL=XS.e=e.\dfrac{CB^2}{eCA}=\dfrac{CB^2}{CA}$. [*Ex. 1.*]

6. *Show that the length of the chord of the auxiliary circle parallel to the minor axis and passing through the focus $=BB'$.*

7. Find the centre of an ellipse, having given a focus, the lengths of the major and minor axes and a point on the ellipse.

[S' is the intersection of the circle with centre P and radius $(AA'-SP)$ with that having S as centre and SS' as radius.] [*Ex. 3*]

8. *Any chord PQ of the circle on the minor axis as diameter meets the circle on the major axis as diameter at p, q. Show that $Qq.Qp=CS^2=Pp.Qp$.*

Let x be the perpendicular distance CO of the chord from the centre C.

Then $OP^2=CP^2-x^2$, and $Op^2=Cp^2-x^2$.

$\therefore \quad Op^2-OP^2=Cp^2-CP^2=CA^2-CB^2=CS^2$.

But $\quad Op^2-OP^2=(Op+OP)(Op-OP)=Qp.\,Pp$.

Proposition VII.

*The middle points of parallel chords of an ellipse lie on a straight line passing through the centre.**

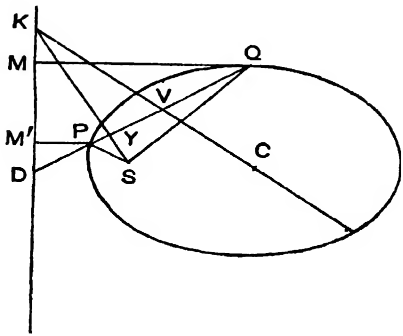

Fig. 27

Let PQ be one of a system of parallel chords of an ellipse and V its middle point. Let S be the focus and C the centre of the ellipse.

It is required to prove that the locus of V is a st. line passing through the centre of the ellipse.

Draw SY perpendicular to PQ to meet the directrix at K. Produce QP to meet the directrix at D. Join PS, QS and draw QM, PM' prependiculars on the directrix.

Since the Δs DPM', DQM are similar,

$$\frac{QD}{PD}=\frac{QM}{PM'}=\frac{QS}{PS}, \text{ whence } \frac{QD^2}{PD^2}=\frac{QS^2}{PS^2}.$$

$$\therefore \quad \frac{QS^2}{QD^2}=\frac{PS^2}{PD^2}=\frac{QS^2-PS^2}{QD^2-PD^2} \quad \cdots \qquad \text{(I)}$$

*C. U. 1913, '15, '22, '24, '28, '31, '33, '35.

But $$QS^2 - PS^2 = (QY^2 + SY^2) - (PY^2 + SY^2)$$
$$= QY^2 - PY^2$$
$$= (QY + PY)(QY - PY)$$
$$= PQ\{(QV + YV) - (PV' - YV)\}$$
$$= 2PQ.YV. \quad \ldots \quad \ldots \quad \text{(II)}$$

Similarly, $$QD^2 - PD^2 = (QD + PD)(QD - PD)$$
$$= 2PQ.DV \quad \ldots \quad \ldots \quad \text{(III)}$$

$\therefore$ from (I), (II) and (III),

$$\frac{PS^2}{PD^2} = \frac{2PQ.YV}{2PQ.DV} = \frac{YV}{DV} \ldots \quad \ldots \quad \text{(IV)}$$

Now the ratio $\frac{SP}{PM'}$ is constant being equal to the eccentricity of the ellipse. ... ... (α)

The ratio $\frac{PM'}{PD}$ depends only on the inclination of PQ to the major axis.* But PQ is drawn in a fixed direction, $\therefore$ $\frac{PM'}{PD}$ is a also constant (for all parallel chords). ... (β)

Hence $\frac{PS}{PD}\left(= \frac{PS}{PM'} \cdot \frac{PM'}{PD}\right)$ is evidently constant from (α) and (β).

$\therefore$ from (IV), $\frac{YV}{DV}$ is a constant ratio for all chords of the system.

*If θ be the angle which PQ makes with the major axis, $\cos\theta = \frac{PM'}{BD}$ = a constant for parallel chords.

But D lies always on the directrix which is a fixed st. line and Y on another fixed st. line *viz.*, the perpendicular drawn from the focus on the system of parallel chords, the two lines intertsecting at K.

$\therefore$ V also must lie on a fixed st. line passing through the common point K. $\left(\because \dfrac{YV}{DV} \text{ is always constant.}\right)$

(By a well-know proposition.)

That is, the locus of the middle point of all chords parallel to PQ is a straight line.

Again, when one of the system of parallel chords passes through C, the centre of the ellipse, it is bisected at that point, from the symmetry of the ellipse; so that C is a point on the locus.

Hence the locus of V is a st. line passing through the centre of the ellipse.

Def. : *The locus of the middle point of a system of parallel chords of an ellipse is called a* **diameter** *of the ellipse.*

Note. We have seen that the middle points of a system of parallel chords of an ellipse lie on a straight line passing through the centre. Diameters are therefore straight lines passing through the centre.

EXERCISES.

1. Having given a diameter of an ellipse, find the system of parallel chords which are bisected by it.

[Let the diameter meet the directrix in K. Join KS, where S is the focus. Chords perpendicular to KS or KS produced form the system of parallel chords.]

2. *Any straight line passing through the centre is a diameter of the ellipse.* [*See Ex. 1.*]

Conjugate Diameters.

Two diameters are said to be conjugate, when each bisects chords parallel to the other.

3. If one diameter of an ellipse bisects chords parallel to a second, the second diameter bisects chords parallel to the first.

[Let a chord PCP' bisect chords parallel to DCD'. Draw $A'Q$ parallel to CD, meeting CP in L and the ellipse in Q. Join AQ meeting CD in H.

CL is parallel to AQ, and AQ is bisected by CD at H.]

N. B. The major axis and the minor axis form a pair of conjugate diameters of an ellipse.

4. Show that the diameter of a system of parallel chords and the perpendicular drawn from the focus on the system of parallel chords intersects on the directrix.

5. *An ellipse being given find its centre, axes, foci and directrices.*
[C. U. 1933, '35.] [*See Prop. IV, Ex. 3.*]

PROPOSITION VIII.

*If any chord QQ' of an ellipse intersect the directrix in D, SD bisects the exterior angle between SQ and SQ'.**

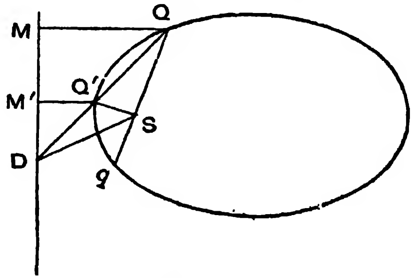

Fig. 28

Let QQ' be any chord of an ellipse with focus S, which meets the directrix in D.

Join DS, QS and $Q'S$ and produce QS to any point q.

It is required to prove that DS bisects the exterior angle $Q'Sq$.

Draw QM and $Q'M'$ perpendiculars to the directrix.

Since the Δs QMD, $Q'M'D$ are similar.

$$\therefore \quad \frac{QD}{Q'D} = \frac{MQ}{M'Q'}. \qquad \ldots \qquad \ldots \qquad (1)$$

But from the definition of the ellipse,

$$SQ = eMQ \text{ and } SQ' = eM'Q'.$$

$$\therefore \quad \frac{MQ}{M'Q'} = \frac{SQ}{S'Q'}. \qquad \ldots \qquad \ldots \qquad (2)$$

$$\therefore \quad \text{from (1) and (2)}, \quad \frac{QS}{Q'S} = \frac{QD}{Q'D}.$$

*C. U. 1919, '23, '2?, '38, '40, '42.

Hence the base QQ' of the $\triangle QQ'S$ is divided externally in the ratio of the sides SQ and $S'Q$.

$\therefore$ SD bisects the vertical angle QSQ' externally.

EXERCISES.

1. *The straight lines joining the vertices of an ellipse to any point P on the curve are produced to meet a directrix in D and D'; shew that DD' subtends a right angle at the corresponding focus.* [C. U. 1919, '40.]

[PS is produced to meet the ellipse again at q.
$D'S$ bisects the $\angle ASP$ and DS bisects the $\angle ASq$.]

2. (*i*) *Find the locus of the pair of points of intersection of the straight lines joining the extremities of a pair of focal chords, as the angle between the focal chords varies. (The locus is the corresponding directrix.)*

(*ii*) *Show also that the intercept on the directrix between each pair of such points subtends a right angle at the focus.*

[*Prop. VIII, Ex. 1, Chapter I.*]

3. *The intercept made on either directrix by straight lines joining the extremities of a focal chord to any point on the curve subtends a right angle at the corresponding focus.* [C. U. 1942.]

[*Chapter I, Prop. VIII, Ex. 2.*]

4. Given the eccentricity, a focus and two points on an ellipse, to construct the curve. [*Ex. 4, Prop. VIII, Chap. I.*]

5. Given a focus and two points on an ellipse, show that the corresponding directrix passes through a fixed point. [In fig. 28, D is the fixed point.]

6. *Having given the focus and three points on an ellipse, to find any number of points on it.* [C. U. 1929.]

[For each pair of points, we can get a point D on the directrix. Find the eccentricity.]

7. *The intercept on either directrix of an ellipse between the straight lines joining the ends of a given chord to a variable point on the ellipse subtends a constant angle at the corresponding focus.*

[*Proceed as in Ex. 1, Prop. VIII, Chapter I.*]

[The constant angle is equal to half the angle subtended by the given chord at the focus.]

8. Any straight line DQ intersects the directrix at D and the ellipse at Q. Take a point p on the curve so that the angle DSp at the corresponding focus is made equal to the angle DSQ. If pS be produced to meet DQ (produced if necessary) at P, shew that P is a point on the ellipse.

(Prove it by reductio ad absurdum).

9. If any chord QQ' meet the directrix in D, then QS and $Q'S$ are equally inclined to DS produced.

10. *Show that no straight line can meet an ellipse in more than two points.* [C. U. 1923, '38.]

The ellipse is a closed curve. [*Prop. I, Ex. 10.*] A straight line will, in general, meet it in two points. Let the straight line $PP'D$ meet the ellipse in P and P', and the directrix in D. Then PS and $P'S$ are equally inclined to DS produced. [*Ex. 9.*]

If it meet the ellipse again at the third point Q, QS, $P'S$ will be similarly equally inclined to DS produced, which is absurd.

11. Show that in an ellipse for any chord QQ', the external bisector of the angle QSQ' and the directrix are concurrent.

12. *Show that the lines joining the extremities of any focal chord to* X *are equally inclined to the axis.*

[Let QSq be any focal chord. Join qX and QX intersecting the ellipse at P. Join PS. Then XS is the bisector of the angle PSq, $\therefore$ $PS=qS$. [*Prop. I, Ex. 12.*]

$\therefore$ the two Δs XPS, XqS are equal in all respects.]

13. The ends P, P' of a focal chord PSP' are joined to any variable point Q, where QP and QP' meet a directrix in p and p'. Prove that the rectangle contained by pX and $p'X$ is a constant ($=SX^2$). [*Ex. 5, Prop. VIII, Chap. I.*]

[The $\Delta pSp'$ contains a right angle at S.] [*Ex. 3.*]

TANGENCY.

PROPOSITION IX.

The tangent to the ellipse at either end of a diameter is parallel to the system of chords bisected by the diameter.

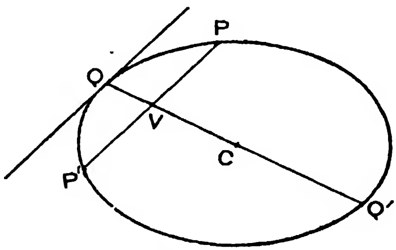

Fig. 29

Let QCQ' be any diameter of the ellipse of which C is the centre. Let PVP' be one of a system parallel chords bisected by QCQ' at V.

It is required to prove that the tangent at Q is parallel to PVP'.

Let the chord PVP' move parallel to itself so that P' coincides with V. In each of its successive positions $PV = P'V$, because QCQ' is the diameter of the system of chords parallel to PVP'.

Hence it is clear that when P' coincides with V, P also coincides with V, and the chord in this limiting position becomes the tangent to the ellipse at Q. Thus the proposition is proved.

N. B. **This method of proof is known as the method of limits.**

Cor. There can be only one tangent to an ellipse at a given point.

*** C. U. 1918, '28.**

EXERCISES.

1. *Draw tangents to an ellipse which will make a given angle with a given straight line.*

[Draw a straight line making, with the given straight line, the given angle. Draw a chord parallel to this straight line.]

2. *Tangents at the vertices of an ellipse are perpendicular to the axis major.* [C. U. 1919.]

[Tangents at the vertices are parallel to the chords bisected by the major axis ; but the major axis bisects chords perpendicular to it, and hence, etc.]

3. *Shew that the straight line joining the points of contact of two parallel tangents to an ellipse is a diameter.* [C. U. 1918, '31.]

[Let the tangents at P and Q be $\parallel$. Join PC and QC, where C is the centre. Then, PC is the diameter of chords parallel to the tangent at P ; QC is the diameter of chords parallel to the tangent at Q. That is, PC and QC bisect the same system.]

$\therefore$ they are parts of the same diameter *viz.* the straight line PQ.

4. Tangents at the extremities of any two diameters of an ellipse form a parallelogram.

5. Draw tangents to an ellipse which shall cut off equal intercepts from the axes.

[Draw a chord equally inclined to the :axes. *See Ex. 1.*]

6. *Draw a tangent to an ellipse parallel to a given st. line, and show that two such tangents are always possible.* [C. U. 1928.]

PROPOSITION X.

The portion of the tangent to an ellipse at any point intercepted between that point and the directrix subtends a right angle at the focus, and conversely.

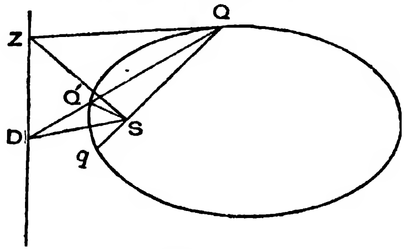

Fig. 30

Let QZ be the tangent to an ellipse at Q meeting the directrix in Z, and S the focus. Join ZS.

It is required to prove that the angle QSZ is a right angle.

Let QQ' be any chord of the ellipse meeting the directrix in D. Join DS, $Q'S$. Produce QS to any point q.

Now DS bisects the exterior angle $Q'Sq$, so that the $\angle Q'SD =$ the $\angle DSq$. [*Prop. VII*]

Let the secant $QQ'D$ be turned about the point Q, so that Q' approaches Q.

Ultimately, when Q' coincides with Q, the secant $QQ'D$ becomes the tangent QZ at Q, D coincides with Z, and the $\angle$s $Q'SD$, qSD coincide with the $\angle$s QSZ, ZSq respectively.

*C. U. 1914, '17, '19, '20, 33, '40, 42.

But the $\angle Q'SD$ is always equal to the $\angle qSD$.

$\therefore$ the $\angle QSZ$ = the $\angle ZSq$.

Hence the $\angle QSZ$ = one right angle.

EXERCISES.

1. **If the tangent at any point P of an ellipse meets the directrices at Z and Z', show that $Z'P$ subtends a right angle at S'.**

2. **The tangent at any point of an ellipse meets the directrices in Z and Z'. The straight line drawn through P perpendicular to the directrices meets them in M and M'. Show that the pts. P, M, S, Z and P, S', M', Z' are concyclic.**

3. ***The tangent at any point P meets the directrix in Z and the latus rectum in H. Show that HS = e.SZ.*** **[C. U. 1942.]**

[Draw PM. Join ZS, PS, MS. From Ex. 2, P, M, S, Z are concyclic ; hence $\angle MPS = 180° - \angle MZS = \angle SZX = \angle ZSH$.

$\because$ $ZX \parallel SH$.

Again, $\angle PMS = \angle PZS$, being in the same circle.

$\therefore$ Δs MPS, ZHS are similar. Hence, etc.]

CONVERSE OF PROPOSITION X.

*If a straight line, drawn from any point on an ellipse to meet the directrix, subtends a right angle at the corresponding focus, then the straight line is the tangent to the ellipse at that point.**

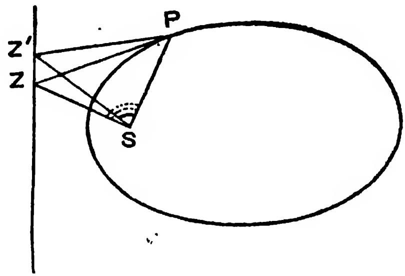

Fig. 31

Let PZ be any straight line drawn from any point P on the ellipse of which S is the focus, to meet the directrix in Z. Let PZ subtend a right angle at the focus.

It is required to prove that PZ is the tangent to the ellipse at P.

If PZ be not the tangent to the ellipse at P, let PZ' be the tangent at P meeting the directrix in Z'.

Now PZ' subtends a right angle at the focus, so that the $\angle PSZ'$ is a right angle.

But by hypothesis, the $\angle PSZ$ is a right angle.

$\therefore$ the $\angle PSZ =$ the $\angle PSZ'$; that is, a part is equal to the whole, which is impossible.

Hence PZ must be the tangent at P.

*C. U. 1917.

EXERCISES.

1. *Draw a tangent to an ellipse at a given point.* [C. U. 1914.]

[If P be the point at which the tangent is to be drawn, join PS. Draw SZ perpendicular to PS meeting the directrix in Z. Then PZ is the tangent required.]

2. Draw a tangent to an ellipse from a point on the directrix.

[If Z be the given point on the directrix, join ZS and draw the focal chord PSp perpendicular to ZS. Then Zp and ZP are the tangents required.]

3. *The tangents at the extremities of a latus rectum intersect the major axis produced on the directrix.* [C. U.]

4. By drawing a tangent at B, the extremity of the minor axis, show that $CS.CX = CA^2$. [C. U. 1918.]

PROPOSITION XI.

*The tangents at the ends of a focal chord of an ellipse intersect on the corresponding directrix.**

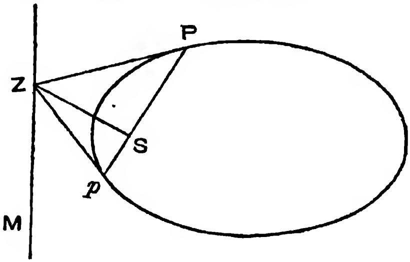

Fig. 32

Let PSp be any focal chord of an ellipse of which S is the focus and MZ the directrix.

Let PZ be the tangent at P meeting the directrix in Z. Join Zp.

It is required to prove that pZ is the tangent at p to the ellipse.

Since PZ is the tangent at P, the $\angle PSZ$ is a right angle. [*Prop. X*]

$\therefore$ the $\angle pSZ$ is also a right angle.

Thus pZ subtends a right angle at the focus S. Hence pZ is the tangent at P. [*Prop. X*]

Note. The locus of the points of intersection of tangents at the ends of focal chords of an ellipse is the corresponding directrix.

Def. : *The line joining the points of contact of two tangents drawn to an ellipse from a point outside it is called the* **chord of contact.**

*C.U. 1909, '15, '18, '19, '34.

EXERCISES.

1. The tangents at the extremities of a focal chord meet the directrix on the focal perpendicular.

2. Prove that tangents at the extremities of the latus rectum intersect where the major axis meets the directrix.

[C. U. 1909, '15, '20, '34.]

[If X be the point of intersection of the major axis with the directrix, XS is the focal perpendicular.

$\therefore$ XL and XL' are tangents at the extremities of the latus rectum LSL'.]

3. The chord of contact of tangents to an ellipse from any point on the directrix passes through the focus.

4. *If the ordinate NP of any point P of an ellipse be produced to meet the tangent at the end of the latus rectum at Q, shew that $QN = SP$, the distance of P from the corresponding focus.* [C. U. 1917, '33.]

[*See Chapter I, Prop. VI, Ex. 13.*]

The $\triangle$s QNX, XLS are similar. $\therefore$ $\dfrac{QN}{XN} = \dfrac{LS}{XS}$.

Hence $QN = \dfrac{LS}{XS} \cdot XN = \dfrac{e.XS}{XS} \cdot XN = e.XN = SP$.

5. Having given the tangents at the extremities of a focal chord, find the corresponding focus.

✓ PROPOSITION XII.*

The tangent at any point on an ellipse makes equal angles with the focal distances of the point.

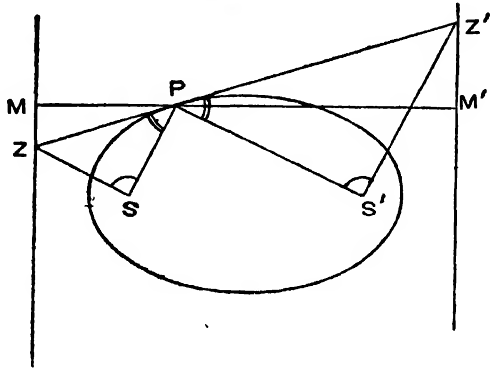

Fig. 33

Let the tangent at any point P of an ellipse meet the directrices in Z and Z'. Join PS, PS'.

It is required to prove that the $\angle SPZ =$ *the* $\angle S'PZ'$.

Through P draw MPM' perpendicular to the directrices, meeting them in M and M'. Join ZS, $Z'S'$.

The Δs PMZ, $PM'Z'$ being similar, $\dfrac{PM}{PM'} = \dfrac{PZ}{PZ'}$, (1)

Again, $PS = ePM$, and $PS' = ePM'$. $\therefore \dfrac{PS}{PS'} = \dfrac{PM}{PM'}$. (2)

Hence, from (1) and (2), $\dfrac{PS}{PS'} = \dfrac{PZ}{PZ'}$.

$$\therefore \frac{PZ^2}{PZ'^2} = \frac{PS^2}{PS'^2} = \frac{PZ^2 - PS^2}{PZ'^2 - PS'^2} = \frac{ZS^2}{Z'S'^2} \text{ i.e., } \frac{PS}{PS'} = \frac{ZS}{Z'S'}.$$

*C. U. 1910, '16, '19, '25, '27, '32, '39, '41, '43.

Thus in the two Δs ZSP, $Z'S'P$, we have $\angle PSZ = \angle PS'Z'$, each being a rt. angle and the sides about the equal angles proportional, $\therefore$ Δs ZSP, $Z'S'P$ are similar.

Hence the $\angle SPZ =$ the $\angle S'PZ'$.

Cor. The tangent at any point P on an ellipse is the external bisector of the angle SPS'. [C. U. 1929.]

EXERCISES.

1. *Show that the bisector of the exterior angle between the focal distances of a point is the tangent at the point.*

Hence construct the tangent at a given point on the ellipse, having given the foci only. [C. U. 1932.]

2. P is any point on an ellipse and T is a point on the major axis produced, so that $\frac{TS'}{ST} = \frac{S'P}{SP}$; shew that PT is the tangent at the point P. [*Ex. 1.*]

3. *The tangent at any point on an ellipse makes a greater angle with the focal distance than with the perpendicular to the directrix.* [C. U. 1916.]

[In fig. 33, since $M'P > S'P$. $\therefore$ $M'Z' < S'Z'$.

$\therefore$ the $\angle M'PZ' <$ the $\angle S'PZ'$, and so on.]

4. The tangent at any point P meets the directrices in Z and Z'. The intercept between the perpendiculars drawn from Z and Z' on SP or $S'P$ (produced) is always a constant.

[Draw $Z'D$ perpendicular to SP produced. The right-angled Δs $PS'Z'$, $PZ'D$ are congruent. $\therefore$ the intercept $SD = SP + S'P = AA'$.]

5. *Having given a focus of an ellipse and the point at which the ellipse touches a given straight line, find the locus of the other focus and that of the centre.* [C. U. 1927.]

[If XY be the given line, P the given pt, on it and S the given focus, the locus of the other focus is a straight line PK passing

through P and having the $\angle XPS$ = the $\angle YPK$. The locus of the centre is a straight line ∥ to PK passing through the mid-point of SP.]

6. *The tangents at the vertices of an ellipse are at right angles to the major axis.* [C. U. 1919.]

[The tangent YAY' at A makes equal angles with the focal distances AS, AS' which are in the same straight line. ∴ the $\angle YAS$ = the $\angle Y'AS'$ = one right angle.]

7. *St, $S't'$ are perpendiculars upon the tangent at any point P of an ellipse. If PN be the ordinate of P, prove that PN bisects the angle tNt'.* [C. U. 1941.]

[The quadrilaterals $PNS't'$, $PNSt$ are concyclic. Join Nt', Nt.

$\angle tNP = \angle tSP = 90° - \angle tPS$, $\angle t'NP = \angle t'S'P = 90° - \angle t'PS'$.

But $\angle tPS = \angle t'PS'$, ∴ $\angle tNP = \angle t'NP$.]

8. Having given a focal chord and the tangents at its extremities, construct an ellipse.

[Draw ZS perpendicular to Pp to find the locus. By Prop. XII, draw $S'P$, pS' to determine the other focus.]

9. *If the perpendicular SO drawn from the focus S on the tangent at any point P meet $P'P$ produced in s, show that (i) $sO = SO$, and (ii) $S's$ is constant.* [C. U. 1910, '39, '43.]

Hence show that the locus of O is the auxiliary circle, i.e., a circle with AA' as diameter.

[$S's = S'P + Ps = S'P + SP$ = the major axis.] [$S's = 2CO = 2CA$.]

Note. The locus of the image 's' of the focus 'S' in the tangent at P is a circle of radius AA' and centre S'.

10. *Having given the foci of an ellipse, and a tangent, show how to find the point of contact.* [C. U. 1925.]

[Draw Ss perpendicular to the given tangent meeting it at y, so that $Sy = sy$. Join sS' intersecting the tangent at P.] [*Ex. 9.*]

Proposition XIII.

*The normal at any point of an ellipse bisects the angle between the focal distances of the point.**

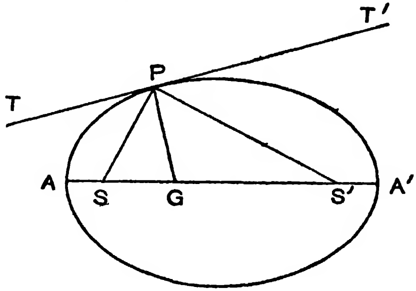

Fig. 34

Let TPT' be the tangent at any point P of an ellipse of which S and S' are the foci.

Draw the normal PG at P meeting the axis in G. Join SP and $S'P$.

It is required to prove that PG bisects the angle SPS'.

Since PG is a normal, the $\angle TPG =$ the $\angle T'PG$, each being a right angle. Again TPT' being the tangent at P, the $\angle TPS =$ the $\angle T'PS'$. [*Prop. XIII.*]

$\therefore$ the $\angle SPG =$ the $\angle S'PG$.

Hence PG bisects the angle SPS' between the focal distances.

EXERCISES.

1. *The normal at any point on an ellipse divides SS' internally in the ratio of the focal distances of the point.*

[The normal is the bisector of the $\angle SPS'$]

*C. U. 1910, '39, '41.

2. Show that the normals at the extremities of the axes pass through the centre of the ellipse. [*Ex. 1.*]

3. If the circle described on the normal PG as diameter intersect the foçal distance of P at U and V, shew that PG bisects UV at right angles.

MISCELLANEOUS EXERCISES.

SEC A. PARABOLA.

1. Any parabola cuts the ordinates (produced if necessary) of any other parabola having the same axis and vertex in a constant ratio.

2. A parabola rolls on an equal parabola, the vertices originally coinciding. Prove that the tangent at the vertex of the rolling parabola always touches a fixed circle.

[The required circle has its centre at S and its radius equal to AS.]

3. Prove that the intercept made on the directrix by two straight lines joining the extremities of a focal chord to any pt. on the parabola subtends a right angle at the focus.

4. A parabola passing through the middle points of the sides of a triangle ABC meets the sides again at a, b, c; show that Aa, Bb, Cc are parallel to one another.

Let D, E, F be the middle points of the sides BC, CA, AB respectively and let BC meet the parabola again at a. Then Da and FE are parallel chords.

$\therefore$ the axis is parallel to the line joining the mid. pts. of Da and FE, which is again parallel to Aa.

5. *If PKQ be any chord of a parabola intersecting the axis at K, prove that $AK^2 = AM.AN$, where M and N are the feet of the ordinates of P and Q respectively.*

From similar $\triangle$s PKM, QKN, we have

$$\frac{KM^2}{KN^2} = \frac{PM^2}{QN^2} = \frac{AM}{AN},$$

$$\text{or, } \left(\frac{AM-AK}{AK-AN}\right)^2 = \frac{AM}{AN}, \text{ whence } AK^2 = AM.\,AN.$$

N. B. *If PQ be a normal chord, $AG^2 = AM.\,AN$;*
if PQ be a focal chord, $AS^2 = AM.\,AN$.

6. *A chord PQ is normal to the parabola at P and subtends a right angle at the vertex ; show that $SQ=3SP$.*

Draw PM, QN ordinates to the points P and Q respectively. From similar Δs PAM, QAN, $\frac{PM^2}{AM^2}=\frac{AN^2}{QN^2}$, whence $16AS^2=AM.AN=AG^2$. [*Ex. 5.*]

$\therefore$ $SP=SG=3AS$. But $AM=SP-AS=2AS$, whence $AN=8AS$. $\therefore$ $SQ=AN+AS=9AS=3SP$.

7. *Show how to draw a pair of tangents to a parabola from a point on the axis.* [C. U. 1921, *See Prop. XI, Ex. 5.*]

8. *Show that the normal at any point P of a parabola meets the axis at a point G which lies on AS produced towards S.* [*Apply Prop. XV.*]

9. *Find the locus of the middle points of focal chords of a parabola.*

Let V be the middle pt. of a focal chord PSp and let the diameter BV intersect the parabola at B. Draw BT tangent to the parabola and BN ordinate to B.

If VM be drawn perpendicular to the axis, Δs BTN, VSM are congruent, $\therefore$ $VM^2=BN^2=4AS.TN=2AS.TN=2AS.SM$.

Hence the locus is a parabola passing through the focus and having the same axis and latus rectum equal to $2AS$.

10. Given a focal chord and tangents at its extremities, find the focus and the directrix of the parabola. [*Prop. XII, Ex. 7.*]

11. In a parabola the normals at the ends of a focal chord meet on the diameter bisecting that chord.

12. A focal chord PSp is bisected at right angles by a line which meets the axis in O ; prove that $Pp=2SO$.

Let Z be the pt. on the directrix at which the tangents at P and p intersect. OV being the perpendicular bisector of Pp, OV is parallel to ZS.

$\therefore$ ZV being parallel to SO, $SO=ZV=\frac{1}{2}Pp$.

13. *The diameter through either end of a focal chord of a parabola bisects the normal chord at the other.*

Let PSp be a focal chord and PGQ the normal chord at P. If pm be perpendicular to the directrix, mS is the focal perpendicular to PGQ. $\therefore$ mp bisects PGQ.

14. *The locus of the foot of the focal perpendicular upon any tangent to a parabola is the tangent at the vertex.* [C. U. 1931.]

Let Y be the pt. of intersection of the tangent at any pt. P with the line MS, where PM is drawn perpendicular on the directrix. Then PY is the perpendicular bisector of MS. $\therefore$ AY is tangent at the vertex.

15. Given the focus and two tangents to a parabola, construct the curve. [*Apply Ex. 14.*]

16. From any point V in a fixed st. line PQ, a straight line $B'V$ is drawn in a fixed direction such that $B'V$ is proportional to $QV.PV$; prove that the locus of B' is a parabola passing through P, Q, and having its axis parallel to $B'V$.

[If O be the middle point of PQ, $PV.QV = PO^2 - OV^2$
$= 4(BS.BO - 4BS.Bv)$, ($\because$ Bv' is the ordinate to the diameter BO)
$= 4BS(BO - Bv) = 4BS.B'V$.]

17. *O is the middle point of a chord PQ of a parabola. The perpendiculars through O to PQ and the axis meet the axis at G and N. Show that GN is equal to the semi-latus rectum.* [C. U. 1926.]

Let O be the middle point of the chord PQ intersecting the axis at t. Draw the diameter through O meeting the curve at B. Let BT be the tangent at B meeting the axis at T and Bn the ordinate to the pt. B.

The $\triangle$s BTn, OtN are congruent.

$\therefore$ $ON^2 = Bn^2 = 4AS.An = 2AS.2An = 2AS.Tn = 2AS.tN$.

[*Prop. XIII.*]

But $ON^2 = tN.NG$. $\therefore$ $tN.NG = 2AS.tN$. Hence $NG = 2AS$.

18. The tangents at the ends of any chord of a parabola intersect on the diameter which bisects the chord, and conversely.

[We know that the st. lines joining the extremities of any two parallel chords of a parabola intersect on their diameter. In this case the two parallel chords become consecutive.]

19. If AR, SY be perpendiculars from the vertex and focus of a parabola upon any tangent, prove that $SY^2 = SY.AR + SA^2$.

20. Given the vertex, a tangent and the latus rectum, construct the parabola.

21. The portion of a diameter intercepted between the tangent at any point of the parabola and the ordinate of the point to the diameter is bisected at its vertex.

Let the tangent at any point P meet BV at R and the tangent at B in T. Draw BK parallel to PR to meet PV at K. Join TK and BP intersecting each other at H. Now H is the mid. pt. of BP and TK, because $PKBT$ is a parallelogram. $\therefore$ TK is a diameter. Hence $RB : BV = TH : HK = 1$. [*Ex. 18.*]

22. The circle circumscribing the triangle formed by any three tangents to a parabola passess through the focus.

23. Describe a parabola to touch four given st. lines.

[*Apply Ex. 22, Ex. 15.*]

24. A parabola rolls on another equal parabola, their vertices being initially coincident; show that the focus of the former describes the directrix of the latter.

Let S, S' be the foci and P the pt. at which the parabolas touch each other. The common tangent at P bisects the $\angle SPS'$. Hence PS' is perpendicular to the directrix of the first parabola and is equal to PS.

25. *Show that there are two points having a given abscissa and that the line joining these points is perpendicular to the axis.* :[Dacca Board.]

(Apply $PN^2 = 4AS.AN$.)

26. *If the double ordinate QQ', be double the length of the latus rectum, shew that it subtends a right angle at the vertex and AQ divides the latus rectum in the ratio of* 1 : 3. [Dacca Board.]

$QN^2 = 4AS.AN = 16AS^2$. $\therefore$ $AN = 4AS$. Hence $QQ'^2 = AQ^2 + A'Q^2 (= 64AS^2.)$ If l be the pt. of intersection of AQ with LS, then $\frac{lS}{AS} = \frac{QN}{AN}$, whence $lS = AS$, and so on.

27. Shew that the semi-latus rectum is a harmonic mean between the segments of any focal chord.

SEC. B. ELLIPSE.

28. The st. lines joining the extremities of any two parallel chords of an ellipse intersect on the diameter which bisects the chord.

Let Pp and Qq be two || chords. Let QP, qp meet the diameter bisecting Pp, Qq at T and t respectively. If V and V' be the middle pts. of Pp, Qq respectively,

$TV : TV' = PV : QV' = pV : qV' = tV : tV'$. $\therefore$ t and T coincide.

29. The tangents at the ends of any chord of an ellipse intersect on the diameter which bisects the chord. [*Ex. 18.*]

30. *O and A are two fixed points. OP is a variable straight line of constant length. On OP a point Q is taken such that $PQ=QA$; show that the locus of Q is an ellipse with O and A for foci. (Note—the point Q is always situated between the points O and P.)* [Dacca Board.]

[$OQ+QA=OP=$ a constant.] [*Apply Prop. V*]

31. If the normal at any point P on an ellipse meet the major axis at G, prove that $SG=e.SP$.

[PG bisects the $\angle SPS'$. $\therefore$ $S'P : SP=S'G : SG$.

$\therefore$ $(S'P+SP) : SP=(S'G+SG) : SG$;

i.e., $AA' : SP=SS' : SG$. Hence $SG=e.SP$.]

32. A chord PQ of an ellipse meets the directrix at D; show that $SP : PD=SQ : QD$.

33. Prove that the external bisector of the angle SPS' is the only st. line passing through P that cannot meet the ellipse again, hence show that it is the tangent at P.

[If XY be the external bisector of the angle SPS', $(SP+S'P)$ is the shortest distance ($=AA'$, the major axis). No other point on XY or on any other straight line passing through P can satisfy this relation and so on.]

34. *In an ellipse if S and S' coincide, the ellipse becomes a circle.* [C. U. 1924.]

[$SP+S'P=2SP=2CA=2CP=$ a constant.] [*Prop. V*]

35. The st. lines joining the foci of an ellipse to the ends of a diameter makes equal angles with the tangents at those ends.

[*Apply Prop. XII.*]

36. Given a focus and the length of the major axis, describe an ellipse touching a given st. line and passing through the given pt. How many solutions does this problem admit of ?

[*Apply Prop. V and Prop. XII.*]

37. Given the distance between the focus and the directrix equal to 3 ins. and the eccentricity equal to $\frac{1}{2}$, determine the positions of A, A', B, B' the extremities of the principal axes. Obtain the length of the 2 principal axes. [C. U. 1913.]

$AS=e.AX.$ $\quad \frac{AS}{AX}=\frac{1}{2}$, or $\frac{XS}{AX}=\frac{3}{2}=\frac{3}{AX}$. $\therefore$ $AX=2$ inches.

Again $A'S=e.AX.$ $\quad\therefore$ $\frac{A'X}{A'S}=2$, or, $1+\frac{XS}{A'S}=2$, whence $A'S=XS=3$. $\therefore$ $A'X=6$ inches, $AA'=e(A'X'+A'X)=4$ inches. $CB^2=SA.SA'=3$ sq. inches.]

38. Given a focus and the length of the major axis, describe an ellipse touching two given st. lines. How many solutions are there ?

39. A piece of paper cut out in the shape of a circle, is folded so that its circumference always passes through a fixed point in the paper ; prove that the creases left in the paper are tangents to an ellipse.

40. (*a*) Prove that the normals at the extremities of the axes all pass through the centre, and conversely.

(*b*) Having given a focus, a tangent and the length of the major axis of an ellipse, find the locus of the other focus.

[The locus is a circle with centre the image of the focus in the given tangent and radius the major axis.]

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41. Show that the two tangents which can be drawn to a parabola from any point on the directrix are at right angles. [C. U. 1927.]

42. In an ellipse show that (i) $BC^2=AS.AS'$;

(ii) $BC^2=SL.CA$.

[*Apply Prop. A*] [C. U. 1926, '30]

43. Given a focus of an ellipse and three tangents, find the other focus. [*Prop. XII.*] [C. U. 1929.]

Let AT, BT be two tangents intersecting at the point T. Make the $\angle ATS=$ the $\angle BTS'$. Then the second focus S' must lie on TS', and so on.

44. The perpendicular SO drawn from the focus S on the tangent at any point P on an ellipse meets $S'P$ produced in S. Show that the locus of S is a circle. [*See Prop. XII, Ex. 9.*]

45. *PN is the ordinate of a parabola ; a straight line drawn parallel to the axis bisects PN and cuts the curve in Q ; NQ meets a line through the vertex A at right angles to the axis in T. Prove that* $3AT=3SN$.

[C. U. 1943.]

[*See Prop. IV, Ex. 11, Chap. I*]

Appendix.

Proposition A.

The square of the ordinate of any point on an ellipse varies as the rectangle under the segments of the axis major made by the ordinate.†

$$(PN^2 : AN.A'N = CB^2 : CA^2)$$

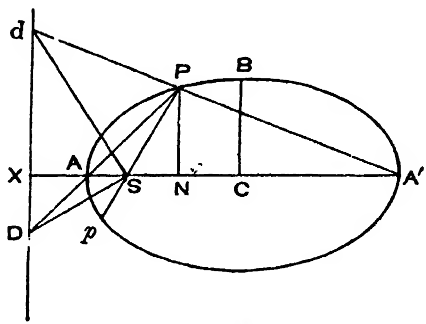

Fig. 35

Let PN be ordinate to any point P on the ellipse of which A and A' are the vertices. Join AP, $A'P$ and produce them to meet the directrix at D and d respectively.

Join DS, dS and produce PS to meet the curve in p.

Now, the Δs DAX, PAN are similar.

$$\therefore \frac{PN}{AN} = \frac{DX}{AX} \quad \ldots \quad \ldots \quad \text{(I)}$$

*Definitions of the 'ordinate' and the 'abscissa' have been given in Chapter I.

†C. U. 1912, '14, '21, '23, '26, '31.

Again the $\triangle$s $dA'X$, $PA'N$ are similar,

$$\therefore \frac{PN}{A'N} = \frac{dX}{A'X}. \quad \ldots \quad \ldots \quad \text{(II)}$$

Hence from (I) and (II), by multiplication,

we have $\dfrac{PN^2}{AN.A'N} = \dfrac{DX \cdot dX}{AX.A'X}$.

Again since, DS and dS bisect the angles pSX and PSX respectively. [*Prop. VIII.*]

$\therefore$ the $\angle DSd$ is a right angle.

$\therefore$ from Geometry, $DX.dX = SX^2$.

Hence $\dfrac{PN^2}{AN.A'N} = \dfrac{SX^2}{AX.A'X}$.

But $\dfrac{SX^2}{AX.A'X}$ is a constant, because S, X, A and A' are fixed points;

$\therefore \dfrac{PN^2}{AN.A'N}$ has the same value for all position of the point P.

Let us find the value of the constant in the particular case, when the point P coincides with the extremity B of the semi-minor axis BC.

Then $\dfrac{PN^2}{AN.A'N}$ becomes $\dfrac{BC^2}{AC^2}$.

$\therefore$ the constant $\dfrac{SX^2}{AX.A'X} = \dfrac{BC^2}{AC^2}$.

Hence $\dfrac{PN^2}{AN.A'N} = \dfrac{CB^2}{CA^2}$, when P is any point on the ellipse.

Alternative Proof

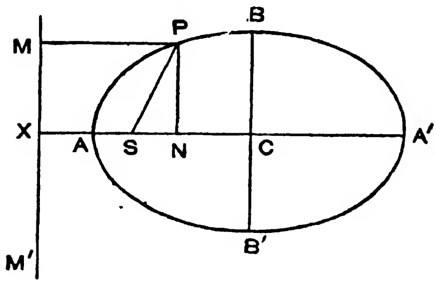

Fig. 36

Let PN be the ordinate to any point P on an ellipse of which S is the focus, MXM' the directrix, AA' the major axis.

Draw the double ordinate BCB' through the centre C of the ellipse.

It is required to prove that $\dfrac{PN^2}{AN.A'N}=\dfrac{CB^2}{CA^2}$.

Join SP and draw PM perpendicular to the directrix. Then

$$PN^2=SP^2-SN^2=(SP+SN)(SP-SN).$$

Now $SP+SN=e(AX+AN)+SN$

$$=AS+eAN+AN-AS=(1+e)AN$$

and $SP-SN=eXN-SN=e(A'X-A'N)-SN$

$$=A'S-eA'N-(A'S-A'N)=(1-e)A'N.$$

$$\therefore\ PN^2=(1+e)(1-e)AN.A'N=(1-e^2)AN.A'N.$$

$\therefore\ \dfrac{PN^2}{AN.A'N}=(1-e^2)$, which is constant for a given ellipse.

We have taken P to be any point on the ellipse, so that the relation $\dfrac{PN^2}{AN.A'N}=(1-e^2)$, is true for any position of P on the curve.

Now in the particular case when P coincides with B so that N coincides with C,

$$\frac{PN^2}{AN.A'N}=\frac{CB^2}{CA^2}. \quad \therefore \text{ the constant } (1-e^2)=\frac{CB^2}{CA^2}.$$

Hence $\dfrac{PN^2}{AN.A'N}=\dfrac{CB^2}{CA^2}$, which is constant.

Note. The ratio of the minor axis to the major one $=\sqrt{1-e^2}$.

EXERCISES.

1. *Show* that $\dfrac{PN^2}{CA^2-CN^2}=\dfrac{CB^2}{CA^2}$; *hence prove that*
$$\frac{CN^2}{CA^2}+\frac{PN^2}{CB^2}=1.$$

[$AN.A'N=(CA-CN)(CA'+CN)=(CA+CN)(CA-CN)$
$=CA^2-CN^2$.]

Since $\dfrac{PN^2}{CA^2-CN^2}=\dfrac{CB^2}{CA^2}$. $\therefore$ $\dfrac{PN^2}{CB^2}=\dfrac{CA^2-CN^2}{CA^2}$.

$\therefore$ $\dfrac{PN^2}{CB^2}=1-\dfrac{CN^2}{CA^2}$. Hence $\dfrac{PN^2}{CB^2}+\dfrac{CN^2}{CA^2}=1$.

2. *Show that the ratio of the minor axis to the major* $=(1-e^2)^{\frac{1}{2}}$: *hence prove that* $e^2=1-\dfrac{CB^2}{CA^2}$. [C. U. 1916, '21, '36.]

3. Enunciate and prove the converse of Prop. III.

Enunciation :

If a point P moves in such a way that $PN^2 : AN.A'N$ is a constant PN being the distance of P from the line joining two fixed points A and A' and N lying between them, the locus of P is an ellipse of which AA' is an axis. [Prove it by the indirect method.]

4. *From any point P on a circle, PN is drawn perpendicular to a diameter meeting it at N. Find the locus of the middle point of PN.*

Let C be the centre of a circle of a diameter AA' and PNP' any chord perpendicular to AA'. If Q be the middle point of the ordinate QN, then $QN^2=\frac{1}{4}PN^2$.

But $PN^2=CP^2-CN^2$.

$\therefore$ $QN^2=\frac{1}{4}(CP^2-CN^2)=\frac{1}{4}(CP+CN)(CP-CN)$
$=\frac{1}{4}(CA+CN)(CA-CN)=\frac{1}{4}(CA'+CN)(CA-CN)$
$=\frac{1}{4}A'N.AN$.

$\therefore$ $\dfrac{QN^2}{AN.AN'}=\frac{1}{4}$, which is constant. Hence the locus of Q is an ellipse.

5. Show that the maximum value of PN is BC. [C. U. 1912, '31.]

We have $\dfrac{PN^2}{AN.A'N}=\dfrac{CB^2}{CA^2}$. $\therefore$ $\dfrac{PN^2}{CA^2-CN^2}=\dfrac{CB^2}{CA^2}$. [*Prop. VIII, Ex. 1.*]

$$\therefore \quad PN^2=\frac{CB^2}{CA^2}(CA^2-CN^2)=CB^2-\left(\frac{CN}{CA}\right)^2\cdot CB^2.$$

Now, PN will be maximum when the negative square term on the right-hand side is zero.

6. *If the ordinates of all points on an ellipse be divided in the same ratio, the locus of the points of division is an ellipse.*

Let the ordinate PN be divided at Q, so that $\dfrac{PQ}{QN}=n$,

whence $\dfrac{QN}{PN}=\dfrac{1}{1+n}=m$ (suppose). $\therefore$ $QN^2=PN^2.m^2$

$$=AN.A'N.\frac{CB^2}{CA^2}\cdot m^2=AN.A'N.k, \text{ where } k \text{ is a constant.}$$

Hence the locus of Q is an ellipse. [*Ex. 3.*]

7. *If the ordinates of all points on an ellipse be produced in the same sense, and in the same ratio, shew that the locus of the extremities of these ordinates, so produced, lie on an ellipse.*

[*Follow the method of Ex. 6.*]

8. *If QM be drawn perpendicular to the minor axis from any point Q on the ellipse, shew that* $\dfrac{QM^2}{BM.MB'}=\dfrac{CA^2}{CB^2}$.

Draw QN, QM perpendiculars on the major and the minor axes respectively.

Now, $BM.B'M=BC^2-CM^2=BC^2-QN^2$

$$=BC^2-\frac{CB^2}{CA^2}(CA^2-CN^2) \qquad [\ Ex.\ 1.\ ]$$

$$=BC^2\left(\frac{CN^2}{CA^2}\right)=QM^2\ \frac{CB^2}{CA^2}.$$

9. If the ordinate PN meet the circle on the major axis AA' as diameter at p, show that $pN : PN=CA : CB$.

$\left[\dfrac{PN^2}{AN.A'N}=\dfrac{CB^2}{CA^2}=\dfrac{PN^2}{pN^2},\right.$ because ApA' is a right-angled triangle. $\left.\right]$

10. P is any point on an ellipse. From A and A' perpendiculars are drawn to AP and $A'P$ to meet at the point Q; shew that the locus of Q is an ellipse of which AA' in the minor axis.

[*Follow Ex. 3, 8.*]

11. Shew that $CP^2 = CB^2 + e^2.CN^2$. Hence deduce that the major axis and the minor axis are the maximum, and minimum central chords of an ellipse.

$$[\ CP^2 = PN^2 + CN^2 = \frac{CB^2}{CA^2}(CA^2 - CN^2) + CN^2$$

$$= CB^2 + CN^2\left(1 - \frac{CB^2}{CA^2}\right) = CB^2 + e^2.CN^2. \qquad [\ Ex.\ 2.\ ]$$

Hence CP is minimum, when CN is zero.

Again, $CP^2 = (1-e^2)CA^2 + e^2.CN^2 = CA^2 - e^2.CA^2 + e^2.CN^2$. [*Ex. 2.*]
$= CA^2 + e^2(CN^2 - CA^2)$.

Hence CP is maximum, when $CN = CA$.]

12. P is any point on an ellipse. AP and $A'Q$ (produced if necessary) meet the minor axis at Q and R respectively; show that $CR.CQ = BC^2$.

[Produce PA, $A'P$ to meet the directrix at D, D' respectively.

The Δs ADX, AQC are similar. The Δs $A'RC$, $A'D'X$ are similar. *See Prop. III.*]

13. Construct an ellipse by the application of Ex. 9.

14. *Shew that the ordinate at any point P increases as it moves from the extremity of the major axis to that of the minor one.* [C. U. 1923.]

$$\frac{CB^2}{CA^2} = \frac{PN^2}{AN.A'N} = \frac{PN^2}{CA^2 - CN^2}, \quad \therefore \quad PN^2 = CB^2 - CN^2 \cdot \frac{CB^2}{CA^2}.$$

Hence PN increases as CN decreases.

15. Prove that AA' (major axis) $= \dfrac{SX - (1-e)AX}{(1-e)}$; hence prove that the other vertex A' and the centre C of a parabola are at infinity.

For an ellipse, $SX = A'X - A'S = A'X - e.A'X = A'X(1-e)$;

$$\therefore \quad A'X = \frac{SX}{1-e}.$$

*The circle described on the major axis of the ellipse as diameter is called the **auxiliary circle**.

But $\quad AA' = A'X - AX = \frac{SX}{1-e} - A'X = \frac{SX - (1-e)AX}{(1-e)}.$

For the parabola, e (eccentricity) $= 1$.

Now AA' (length of the axis of a parabola) $\to \infty$ as $e \to 1$

$$\therefore \quad CA = \infty.$$

Hence the centre C and the other vertex A' of a parabola are at infinity.

Note. In a parabola, the other focus S' is at infinity.

Proposition B.

*The ellipse is symmetrical with respect to its minor axis.**

(An alternative proof.)

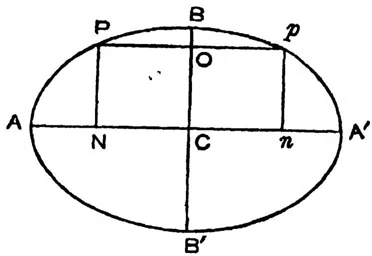

Fig. 37

Let AA' be the major axis of an ellipse of which C is the centre.

Through C draw a chord BCB' perpendicular to the major axis, meeting the ellipse in B and B'.

It is required to prove that the ellipse is symmetrical with respect to the minor axis BCB'.

Let Pp be any chord at right angles to BCB' meeting it in O. Draw PN, pn ordinates to the points P and p respectively.

*C. U. 1925. '28, '41.

Now, $\dfrac{PN^2}{AN.A'N} = \dfrac{CB^2}{CA^2}$ [*Proposition II.*]

Again, $\dfrac{pn^2}{An.A'n} = \dfrac{CB^2}{CA^2}$ [*Prop. A.*]

$\therefore \dfrac{PN^2}{AN.A'N} = \dfrac{pn^2}{An.A'n}$, each being equal to $\dfrac{CB^2}{CA^2}$

But $AN.A'n = (CA + CN)(CA - CN) = CA^2 - CN^2$.

Similarly $An.A'n = CA'^2 - Cn^2$.

$$\therefore \frac{PN^2}{CA^2 - CN^2} = \frac{pn^2}{CA'^2 - Cn^2} \tag{1}$$

Since $PNnp$ is a rectangle by construction, $PN = pn$ (2)

$\therefore$ from (1) and (2), $CA^2 - CN^2 = CA'^2 - Cn^2$.

Since $CA^2 = CA'^2$, $\therefore$ $CN^2 = Cn^2$. Hence $CN = Cn$.

But $CN = PO$ and $Cn = pO$. [$\because$ the quadrilateral $PNCO$, $pnCO$ are rectangles.]

$\therefore$ $PO = pO$. That is, Pp is bisected at O by BCB'.

But Pp is any chord of the ellipse at right angles to BCB',

$\therefore$ BCB' bisects every chord of the ellipse at right angles to it.

Hence the ellipse is symmetrical with respect to the minor axis BCB'.

CO-ORDINATE GEOMETRY

CHAPTER I.

CO-ORDINATES.

Co-ordinates—The position of a point in a plane can be determined with the help of two given intersecting straight lines in the plane which divide it into four parts. Let XOX' and YOY' be two fixed straight lines intersecting at right angles.

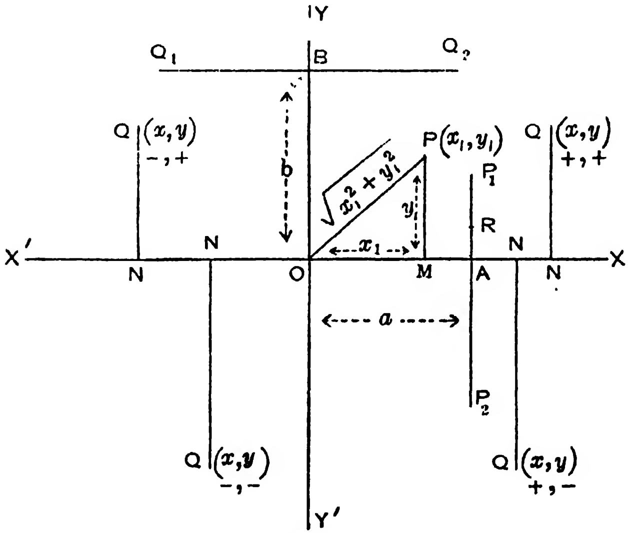

Fig. 1

Take any point Q in the plane. From Q draw QN perpendicular to XOX'. Then it is evident that the position of Q will be known, if ON and NQ are known.

ON is called the **abscissa** or the ***x*-co-ordinate** and NQ is called the **ordinate** or the ***y*-co-ordinate** of the point Q. The line XOX' is called the **axis of *x***, the line YOY' the **axis of *y*** and the two lines XOX' and YOY' are together called the **axes of co-ordinates.** The point O is called the **origin.**

The four parts *viz.* XOY, YOX', $X'OY'$ and $Y'OX$ into which the plane is divided by the axes of co-ordinates are respectively called the first, second, third and fourth quadrants.

The **convention** adopted for the signs of co-ordinates of any point in a plane, containing two axes may be stated thus :

Any distance measured parallel to the axis of x is taken to be positive, if it lies to the right side of the axis of y and negative, if it lies to the left. Any distance measured parallel to the axis of y is taken to be positive, if it lies above the axis of x and negative, if it be below the axis of y.

Hence (1) for any point in the first quadrant XOY, both x and y co-ordinates are positive.

(2) For any point in the second quadrant YOX' the y-co-ordinate is positive and the x-co-ordinate is negative.

(3) For any point in the third quadrant $X'OY'$, both x and y-co-ordinates are negative.

(4) For any point in the fourth quadrant $Y'OX$, the x-co-ordinate is positive and the y-co-ordinate is negative.

When the axes of co-ordinates XOX', YOY' are inclined to each other at an angle other than a right angle, they are called **oblique axes.** The convention regarding signs in the case of both the oblique and the rectangular axes is exactly the same. In both cases the x-co-ordinate of any point on the y-axis is zero, the y-co-ordinates of any point on the x-axis is zero, so that the co-ordinates of the origin O are $(0, 0)$.

This system of co-ordinates (*viz.*, the rectangular co-ordinates and the oblique co-ordinates) is known as the Cartesian system of co-ordinates, because it was first introduced by the philosopher Descartes.

We shall assume the axes to be ***rectangular***, if not otherwise mentioned.

The x and y co-ordinates of a point will usually be denoted by symbols like x, y respectively.

Thus, if a point P lies in the first quardrant and if its geometrical distances from the axes XOX' and YOY' are respectively b and a, the fact will be symbolically stated by writing '$P(a, b)$'. If the point lies in the second quadrant it is $P(-a, b)$.

1. ***To find the distance of any pt. (x_1, y_1) from the origin.***

Let P be the pt. (x_1, y_1).

In fig. 1, draw PM perpendicular to the x-axis.

Join OP. Then, $OP^2 = OM^2 + PM^2$

$$= x_1^2 + y_1^2,$$

whence $PO = \sqrt{x_1^2 + y_1^2}$.

2. *To find the distance between two points whose co-ordinates are given.*

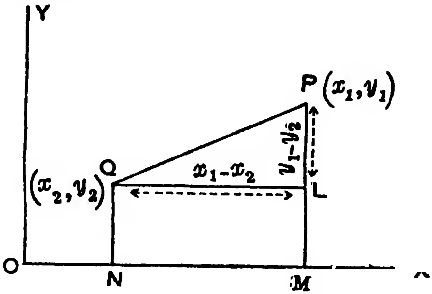

Fig. 2

Let P and Q be two given points, and let their co-ordinates be (x_1, y_1) and (x_2, y_2) respectively with reference to the rectangular system OX, OY.

Draw PM, QN perpendiculars to OX and QL perpendicular to PM, so that $OM = x_1$, $ON = x_2$, $PM = y_1$, $QN = y_2$.

Now in the right-angled $\triangle PQL$, $\quad PQ^2 = PL^2 + QL^2$.

But $\quad QL = NM = OM - ON = x_1 - x_2$,

and $\quad PL = PM - LM = PM - QN = y_1 - y_2$.

Hence $\quad PQ^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$,

or, $\quad PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

This formula holds good whatever be the values of x_1, y_1, x_2, y_2, positive or negative.

Note. The distance of P from the origin (*i.e.*, *Art. I*) can be obtained by putting $x_2 = 0 = y_2$, in the above result, so that Q coincides with the origin.

Ex. 1. Find the distance between the origin and the point (5, 12).

The co-ordinates of the origin are (0, 0).

$\therefore$ The reqd. distance $= \sqrt{5^2 + 12^2} = 13$.

Ex. 2. Find the distance between the points $P(26, 10)$ and $Q(2, 3)$.

$PQ^2 = (26 - 2)^2 + (10 - 3)^2 = 24^2 + 7^2 = 625 = 25^2$. $\quad \therefore \; PQ = 25$.

Ex. 3. Show that the distance between $P(a, -b)$ and $Q(-a, b)$ is $2\sqrt{a^2+b^2}$.

$$PQ^2 = \{a-(-a)\}^2 + (-b-b)^2 = (2a)^2 + (-2b)^2 = 4(a^2+b^2).$$

$$\therefore \quad PQ = 2\sqrt{a^2+b^2}.$$

Ex. 4. If the point (x, y) be always equidistant from the points $(7, -3)$ and $(-5, 4)$, show that $14y - 24x + 17 = 0$.

The distance between (x, y) and $(7, -3) = \sqrt{(x-7)^2+(y+3)^2}$. The distance between (x, y) and $(-5, 4) = \sqrt{(x+5)^2+(y-4)^2}$. Hence, by the question, we have $(x-7)^2+(y+3)^2 = (x+5)^2+(y-4)^2$,

$$\begin{aligned} \text{or,} \quad & x^2+y^2-14x+6y+58 = x^2+y^2+10x-8y+41, \\ \text{or,} \quad & -14x+6y+58 = 10x-8y+41, \\ \text{or,} \quad & -24x+14y+17 = 0. \\ \therefore \quad & 14y-24x+17 = 0. \end{aligned}$$

Ex. 5. Prove that the points $A(2, 4)$, $B(2, 6)$ and $C(2+\sqrt{3}, 5)$ are the vertices of an equilateral triangle whose side is 2.

$$\begin{aligned} AB^2 &= (2-2)^2+(6-4)^2 = 4. & \therefore \ AB &= 2. \\ BC^2 &= (2+\sqrt{3}-2)^2+(5-6)^2 = 3+1 = 4. & \therefore \ BC &= 2. \\ CA^2 &= (2+\sqrt{3}-2)^2+(5-4)^2 = 3+1 = 4. & \therefore \ CA &= 2. \end{aligned}$$

Hence the result.

EXAMPLES I.

1. Find the distance between the following pairs of points :

(i) $(0, 0)$, $(a\cos\phi, a\sin\phi)$; (ii) (a, b), (c, d) ;

(iii) $(3, -2)$, $(-1, 1)$; (iv) $(a\cos\theta, a\sin\theta)$, $(a\cos\phi, a\sin\phi)$;

(v) $(11, 3)$, $(3, -3)$; (vi) $(-15, -25)$; $(-5, -10)$.

2. The distance between the points $(x, 5)$ and $(8, 3)$ is 2. Find the value of x.

3. The square of the distance between the points $(5, 10)$ and $(10, y)$ is 50. Find the ordinate of the unknown pt.

4. If the point (x, y) be equidistant from the two points $(3, 5)$ and $(2, -3)$, then will $2x+16y=21$.

5. Find the co-ordinates of the point which is equidistant from $(5, -6)$, $(3, -4)$ and $(1, 2)$.

CHAPTER II.

THE STRAIGHT LINE.

3. The **locus** of a point is the path traced out by it as it moves satisfying some given condition.

The relation or condition to be satisfied by the moving point may be expressed in terms of its co-ordinates. The **equation** to the locus (*i.e.*, the curve or the st. line) is the relation which exists between the co-ordinates of any point on the locus and which is never satisfied by the co-ordinates of any point outside it.

Suppose a point (x, y) moves in such a way that its ordinate is always equal to m times its abscissa plus c *i.e.*, $y=mx+c$, so that $y=mx+c$ is the equation to the locus or the path traced out by the moving point.

The x-axis is the locus of points for which $y=0$. Hence $y=0$ is the condition satisfied by all points on the x-axis, so that $\boldsymbol{y=0}$ is the equation to the x-axis. Similarly $\boldsymbol{x=0}$ is the equation to the y-axis.

4. *To find the equation to a st. line parallel to one of the co-ordinate axes.*

Let P_1AP_2 (in fig. 1) be any st. line parallel to the y-axis cutting the x-axis at A, so that $OA=a$. If R be any point (x, y) on the st. line P_1P_2, the abscissa of R is always equal to a. Hence $\boldsymbol{x=a}$ is the equation to the st. line P_1AP_2.

Similarly the equation to the st. line Q_1BQ_2 (in fig. 1) parallel to the x-axis and at a distance b from it, is given by $\boldsymbol{y=b}$.

5. ***To find the equation to the st. line which is inclined to the axis of x at a given angle and cuts off a given intercept from the y-axis.***

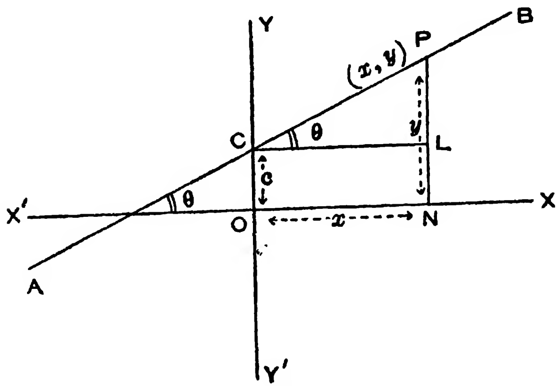

Fig. 3

Let AB be any st. line, inclined at an angle θ to the x-axis and cutting off an intercept OC $(=c)$ from the y-axis. Take any point $P(x, y)$ on it. From P draw PN perpendicular to the x-axis and from C draw CL perpendicular to PN, so that the $\angle PCL=\theta$.

$$\begin{aligned} \text{Now}\quad y=PN=PL+LN&=CL\tan\theta+CO\\ &=ON\tan\theta+c=x\tan\theta+c,\end{aligned}$$

i.e., $y=mx+c$, where $\tan\theta=m$.

Hence $\boldsymbol{y=mx+c}$ is the required equation.

Note 1. Thus 'm' is the tangent of the angle which the st. line makes with the positive direction of the axis of x and 'c' is the intercept made by the line on the y-axis.

Cor. *Any equation of the first degree in x and y always represents a straight line.*

Note 2. In order to obtain the 'm' of a line, $Ax+By+C=0$ take the term containing y to one side and all other terms to the other. Then divide both sides by the coefficient of y. The coefficient of x now gives the 'm' and the constant term the 'c' for the st. line.

Consider the equation $Ax+By+C=0$.

Transposing, we have $By=-Ax-C$, *i.e.*, $y=-\dfrac{A}{B}x-\dfrac{C}{B}$.

Hence we can interpret it as the equation to a st. line whose intercept on the y-axis is $-\dfrac{C}{B}$ and which makes an angle θ with the x-axis, such that $\tan\theta=-\dfrac{A}{B}$.

Note 3. When m is zero, the equation $y=mx+c$ is reduced to $y=0x+c$, *i.e.*, $y=c$, whence the line is parallel to the x-axis. When $c=0$, the equation becomes $y=mx$, *i.e.*, the st. line passes through the origin.

Note 4. When two lines are parallel, their m's are equal, because they make the same angle with the axis of x.

Ex. 1. Find the equation of a st. line which makes an angle $\tan^{-1}\frac{2}{3}$ with the positive direction of the axis of x, and cuts off an intercept $\frac{1}{3}$ from the y-axis.

Let the equation of the line be $y=mx+c$.

Here $m=\frac{2}{3}$ and $c=\frac{1}{3}$. Hence the required equation is $y=\frac{2}{3}x+\frac{1}{3}$, *i.e.*, $3y=2x+1$, or, $3y-2x-1=0$.

Ex. 2. Find inclination of the st. line $5x+3y+2=0$ to the axis of x and determine its intercept on y-axis.

$$5x+3y+2=0, \text{ or, } 3y=-5x-2,$$
$$\text{or, } y=-\tfrac{5}{3}x-\tfrac{2}{3}.$$

Hence the required angle is $\tan^{-1}(-\frac{5}{3})$, and the line cuts off an intercept $\frac{2}{3}$ from the negative side of the y-axis.

6. *To find the equation of the straight line passing through a given point* (x_1, y_1) *and inclined at an angle* $\tan^{-1}m$ *to the x-axis.*

Since m is given, we can assume the equation, to the line to be $y=mx+c$, where c is unknown. Hence, let the equation of the st. line be $y=mx+c$. ... ... (i)

Since it passes through (x_1, y_1),

$$y_1=mx_1+c. \quad \ldots \quad \ldots \text{(ii)}$$

Subtracting (ii) from (i), we have $\boldsymbol{y-y_1=m(x-x_1)}$, which is the required equation.

7. *To find the equation of the straight line passing through the two points* (x_1, y_1) *and* (x_2, y_2).

Let the equation of the line be $y=mx+c$, ... (i)

where m and c are both unknown.

Then, the points (x_1, y_1) and (x_2, y_2) being on (i),

$$y_1=mx_1+c \quad \ldots \quad \ldots \text{(ii)}$$

$$y_2=mx_2+c \quad \ldots \quad \ldots \text{(iii)}$$

Subtracting (ii) from (iii) and (ii) from (i), we have,

$$y_2-y_1=m(x_2-x_1) \quad \ldots \quad \ldots \text{(iv)}$$

$$\text{and} \quad y-y_1=m(x-x_1). \quad \ldots \quad \ldots \text{(v)}$$

Dividing (v) by (iv), we have the required equation

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \quad \text{i.e.,} \quad \boldsymbol{y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)}.$$

8. *To find the equation of a straight line, having given the intercepts made by it on the axes of co-ordinates.*

Let the st. line AB cut the axes of co-ordinates at A and B, so that the intercept $OA=a$ and $OB=b$.

Take any point $P(x, y)$ on it, and draw PN perpendicular to the x-axis, so that $PN=y$ and $ON=x$.

Now the Δs PNA, ABO are similar.

$$\therefore \quad \frac{ON}{OA} = \frac{x}{a} = \frac{BP}{AB} \quad \cdots \quad \text{(i)}$$

$$\text{and} \quad \frac{PN}{OB} = \frac{y}{b} = \frac{PA}{AB} \quad \cdots \quad \text{(ii)}$$

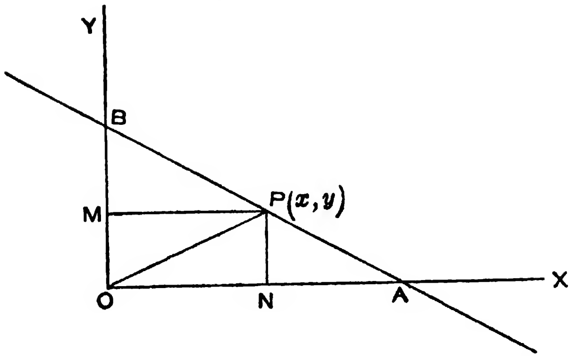

Fig. 4

Adding (i) and (ii), we have

$$\frac{x}{a} + \frac{y}{b} = \frac{BP}{AB} + \frac{PA}{AB} = \frac{AB}{AB} = 1.$$

Hence the required equation is $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

Otherwise : Join OP. Draw PN, PM perpendiculars on the axes of co-ordinates.

The area of the ΔAOB = the area of the ΔOPA + the area of the ΔOPB,

or, $\frac{1}{2}OA.OB = \frac{1}{2}OA.PN + \frac{1}{2}OB.PM$,

or, $\frac{1}{2}ab = \frac{1}{2}ay + \frac{1}{2}bx$.

Dividing both sides by $\frac{1}{2}ab$, we have $1 = \dfrac{y}{b} + \dfrac{x}{a}$

$$\text{or,} \quad \frac{x}{a} + \frac{y}{b} = 1.$$

Ex. 1. Find the intercepts made by any st. line $Ax+By+C=0$ on the axes of co-ordinates.

The given equation is

$$Ax+By+C=0, \quad \text{or,} \quad \frac{A}{C}x+\frac{B}{C}y+1=0,$$

$$\text{or,} \quad \frac{x}{\frac{C}{A}}+\frac{y}{\frac{C}{B}}+1=0, \quad \text{or,} \quad \frac{x}{-\frac{C}{A}}+\frac{y}{-\frac{C}{B}}=1.$$

Hence the intercepts made by the st. line on the axes of x and y are $-\frac{C}{A}$ and $-\frac{C}{B}$ respectively.

Note. Since the equation of the x-axis is $y=0$, the intercept made by the st. line $Ax+By+C=0$ on the x-axis can be obtained by putting $y=0$ in $Ax+By+C=0$; whence $Ax+B.0+C=0$, *i.e.*, $x=-\frac{C}{A}$ (the intercept on the x-axis). Similarly putting $x=0$ in $Ax+By+C=0$, the intercept on the y-axis $=-\frac{C}{B}$.

Ex. 2. Find the equation of the straight line which passes through the points (1, 2) and (2, 1). Find also the length of the st. line intercepted between the axes. [C. U. 1936.]

Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1.$ ... (i)

Since (i) passes through the pts. (1, 2) and (2, 1), we have

$$\frac{1}{a}+\frac{2}{b}=1 \quad \cdots \text{(ii)} \quad \text{and} \quad \frac{2}{a}+\frac{1}{b}=1 \quad \ldots \text{(iii)}$$

Solving (ii) and (iii) for a and b, we have $a=b=3$.

Hence the equation of the line is $\frac{x}{3}+\frac{y}{3}=1.$

$\therefore$ The intercepts made by the line on the axes of co-ordinates are 3 and 3. Hence, the length of the st. line intercepted between the axes $\sqrt{3^2+3^2}=\sqrt{18}=3\sqrt{2}.$

Otherwise : Let the equation of the line be $y=mx+c$.

Then we have $\quad 2=m+c \quad \ldots \quad \ldots \quad$ (i)

and $\quad 1=2m+c \quad \ldots \quad \ldots \quad$ (ii)

Solving (i) and (ii) for m and c, we have $m=-1$ and $c=3$.

Hence the equation is $y=-x+3$. *i.e.*, $\frac{x}{3}+\frac{y}{3}=1$, and so on.

Ex. 3. **Find the equation of the straight line passing through the point (11, −7) and making equal intercepts on the axes of co-ordinates.**

Let the equation of the straight line be $\dfrac{x}{a}+\dfrac{y}{b}=1$.

Since the intercepts are equal, $a=b$.

$\therefore$ the equation $\dfrac{x}{a}+\dfrac{y}{b}=1$ becomes $\dfrac{x}{a}+\dfrac{y}{a}=1$, or, $x+y=a$.

Since the line passes through (11, −7), we must have

$11-7=a$, or, $a=4$, so that the required equation is $x+y=4$.

Ex. 4. **A straight line moves in such a way that the sum of the reciprocals of its intercepts on the axes of co-ordinates is always constant. Show that it passes through a fixed point.**

Let the equation of the straight line be $\dfrac{x}{a}+\dfrac{x}{b}=1$. ... (i)

By the condition of the problem, we have

$$\frac{1}{a}+\frac{1}{b}=\text{a constant}=\frac{1}{k}, \quad \text{(suppose).}$$

$$\therefore \quad \frac{k}{a}+\frac{k}{b}=1 \quad \cdots \quad \cdots \quad \cdots \quad \text{(ii)}$$

from (i) and (ii) it is evident that the pt. (k, k) is on the st. line (i).

Hence, we find that the straight line (i) always passes through the fixed point (k, k), whatever be the values of a and b.

9. *To find the co-ordinates of the point of intersection of two given straight lines.*

Let the equations of the given lines be

$$Ax+By+C=0 \quad \cdots \quad \cdots \quad \text{(i)}$$
$$ax+by+k=0 \quad \cdots \quad \cdots \quad \text{(ii)}$$

If the co-ordinates of the common point of intersection be (α, β), we have

$$A\alpha+B\beta+C=0 \quad \cdots \quad \cdots \quad \text{(iii)}$$
$$a\alpha+b\beta+k=0 \quad \cdots \quad \cdots \quad \text{(iv)}$$

By the method of cross-multiplication from (iii) and (iv),

we have $$\frac{\alpha}{Bk-Cb}=\frac{\beta}{Ca-Ak}=\frac{1}{Ab-Ba},$$

$$\therefore \quad \alpha=\frac{Bk-Cb}{Ab-Ba} \text{ and } \beta=\frac{Ca-Ak}{Ab-Ba}.$$

Note 1. The co-ordinates (α, β) of the point of intersection are infinite, if $Ab-Ba=0$,

i.e., $\frac{A}{B}=\frac{a}{b}$, whence 'm' of (i) must be equal to the 'm' of (ii).

This gives the condition that two st. lines may be parallel.

Note 2. Of two st. lines at right angles, if one be inclined to the axes of x at an angle θ, the other must be inclined at an angle $(\theta \pm 90°)$ to the x-axis, so that 'm' of the first line is $\tan\theta=m$, say (i), and 'm' of the second st. line is $\tan(\theta\pm 90°)=-\cot\theta=m_2$, say (ii). Hence $m_1m_2=\tan\theta.(-\cot\theta)=-1$.

Thus 'm' of the first line × 'm' of the second line = − 1. This gives the condition that two st. lines may be perpendicular.

EXAMPLES II.

1. Find the equation of the straight line which cuts off an intercept -3 from the y-axis and is inclined at an angle of 45° to the positive direction of the x-axis.

Draw a sketch of the straight line and show from geometrical consideration that this straight line is at right angles to the straight line $x+y=2$. [C. U. 1939.]

[The intercepts made by the first line are 3 and -3 and those of the second line are 2 and 2.]

Find the equation to the st. line.

2. Cutting off intercepts -7 and 8 from the axes.

3. Cutting off intercepts -5 and -4 from the axes.

4. Find the equation to the straight line which passes through the point (15, 13) and makes intercepts on the axes equal in magnitude but opposite in sign.

5. Find the equation to the straight line cutting off an intercept -5 from the y-axis and inclined to the x-axis at an angle, $\sin^{-1}\frac{5}{13}$.

Find the equation of the straight line passing through the following pairs of points :

6. (2, 3) and (9, 15). 7. $(a, 0)$ and $(0, b)$.

8. (0, 0) and (c, d). 9. $(a \cos \alpha, b \sin \alpha)$ and $(a \cos \beta, b \sin \beta)$.

10. Find the equation of the straight lines passing through points (4, 5) and (2, 3) and calculate the length of the straight line intercepted between the axes.

11. Determine the co-ordinates of the points of intersection of $ax+by-c=0$ with the axes of co-ordinates and calculate its length intercepted between the axes of co-ordinates.

Find the co-ordinates of the points of intersection of the following pairs of straight lines :

12. (i) $2y-x-5=0$ and $y-x-1=0$.

(ii) $\frac{x}{3}+\frac{y}{2}=1$ and $\frac{x}{2}+\frac{y}{3}=1$. [C. U. 1943.]

13. $x-y-2=0$ and $10x-y-65=0$.

14. Show that the straight line $2x-y-1=0$ passes through the point of intersection of $2y-x-1=0$ and $17x-3y-14=0$.

15. Show that the straight lines $y-x+1=0$, $2y-x-1=0$, and $y-7x+19=0$ are concurrent.

[Solve any two equations and substitute the values of x, y in the remaining equation.]

16. Determine the value of m for which the straight lines $\frac{x}{p}+\frac{y}{q}=1$, $\frac{x}{q}+\frac{y}{p}=1$, and $y=mx$ will be concurrent.

17. Show that the straight line $3x+4y-7=0$ is parallel to the straight line $6x+8y+100=0$.

18. Find the equation of a straight line passing through the point (6, 5) and parallel to the line $y-2x+10=0$.

19. Show that the following pairs of st. lines are at right angles :

(i) $3x+4y+7=0$ and $4x-3y+11=0$.

(ii) $4x+7y+13=0$ and $7x-4y+m=0$.

(iii) $ax+by+c=0$ and $ay-bx+k=0$.

20. Find the equation to the st. line which passes through the point $(4, -5)$ and is perpendicular to the st. line $3x+4y=17$.

CHAPTER III.

THE CIRCLE.

10. *To find the equation of a circle of given radius with the origin* (0, 0) *as centre.*

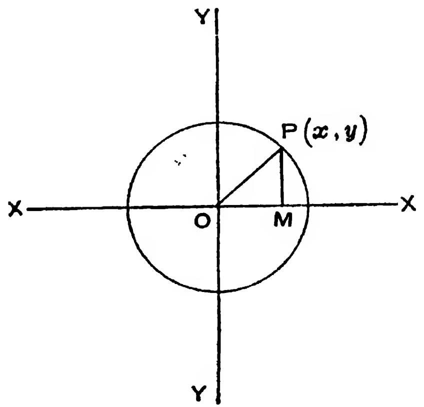

Fig. 5.

Let XOX', YOY' be two rectangular axes. Let $P(x, y)$ be any point on the circumference of the circle, whose centre is the origin O and radius is a.

Draw PM perpendicular to the x-axis. Join OP. Then in the right-angled $\triangle OPM$,

$$OP^2 = PM^2 + OM^2, \quad i.e., \ a^2 = x^2 + y^2.$$

$$[\because \ OM = x, \ PM = y, \text{ and } OP = a.]$$

Hence the required equation is $\boldsymbol{x^2 + y^2 = a^2}$.

11. *To find the equation of a circle of given radius with any point* (h, k) as centre.

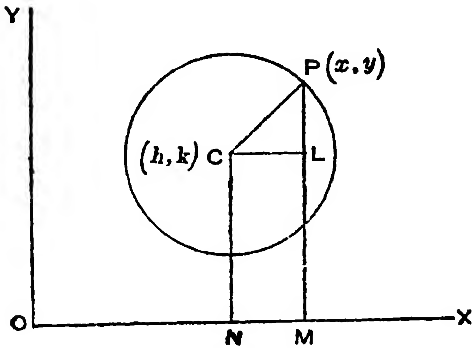

Fig. 6.

Let OX and OY be two rectangular axes. Take any point $P(x, y)$ on the circumference of the circle, whose centre is the point $C(h, k)$ and radius is a.

Draw PM, CN perpendiculars to OX, and CL perpendicular to PM.

In the right-angled $\triangle CPL$, $CL^2 + PL^2 = CP^2$.

But $\quad CL = NM = OM - ON = x - h,$

$PL = PM - LM = PM - CN = y - k$, and $CP = a$.

$\therefore \quad (x-h)^2 + (y-k)^2 = a^2 \quad \ldots \quad \ldots \quad$ (i)

This is the required equation.

12. *To prove that* $x^2+y^2+2gx+2fy+c=0$ *always represents a circle for all values of* g, f *and* c, *and to determine its radius and centre.*

The given equation may be written in the form

$$(x^2+2gx+g^2)+(y^2+2fy+f^2)-g^2-f^2+c=0,$$

$$i.e.,\ (x+g)^2+(y+f)^2=g^2+f^2-c,$$

$$i.e.,\ (x+g)^2+(y+f)^2=\{\sqrt{g^2+f^2-c}\}^2.$$

Comparing this equation with equation (i) of Art. 10, we find both the equations are the same, if

$h=-g=$ the x-co-ordinate of the centre of the circle,

$k=-f=$ the y-co-ordinate of the centre of the circle,

and $a=\sqrt{g^2+f^2-c}=$ the radius of the circle.

Since $\sqrt{g^2+f^2-c^2}$ is the radius of the circle,

if g^2+f^2-c be positive, the circle is real ;

if g^2+f^2-c be negative, the circle is imaginary,

because the radius is imaginary ;

if $g^2+f^2-c=0$, the radius is zero, so that it represents a point circle or a circle of zero radius.

Note. The equation of a circle must satisfy two conditions, *viz.*,

(1) the coefficients of x^2 and y^2 must be equal.

(2) the coefficient of xy must be zero.

The general equation of the second degree *viz.*,

$ax^2+by^2+2hxy+2gx+2fy+c=0$, represents a circle if $a=b$ and $h=0$.

13. *To find the co-ordinates of the points of intersection of a circle and a straight line.*

Let the equations of the circle and the straight line be $x^2+y^2=a^2$ ... (i) and $y=mx+c$ ... (ii), respectively.

To find the co-ordinates of the points of intersection of the circle and the st. line, we have to solve the two equations (i) and (ii).

Substituting the value of y from (ii) in (i), we have

$$x^2+(mx+c)^2=a^2,$$

i.e., $$x^2(1+m^2)+2mcx+c^2-a^2=0. \qquad \dots \text{(iii)}$$

The roots of the quadratic equation (iii) give the x-co-ordinates of the two points of intersection of (i) and (ii). When these two roots are substituted in (ii), we get the y-co-ordinates of the two points of intersection of the circle and the st. line. The two points of intersection may be real or imaginary according as the roots of (iii) are real or imaginary. The two points of intersection of (i) and (ii) will be coincident or the st. line $y=mx+c$ will be a tangent to the circle $x^2+y^2=a^2$, if the roots of (iii) be *real* and *equal*,

i.e. if $4m^2c^2-4(1+m^2)(c^2-a^2)=0$,

or, $m^2c^2-(c^2-a^2+m^2c^2-m^2a^2)=0$,

or, $(1+m^2)a^2=c^2$ $\qquad \therefore \quad c=\pm\sqrt{1+m^2}$.

Substituting the values of c in (i), we find that the straight lines $y=mx\pm a\sqrt{1+m^2}$ are each a tangent to the circle $x^2+y^2=a^2$.

Ex. 2. Obtain the equation of the circle whose centre is the point (2, 3) and which passes through the point (5, 7). [C. U. 1937.]

Radius of the circle = the distance between the centre (2, 3) and the point (5, 7) = $\sqrt{(5-2)^2+(7-3)^2}=\sqrt{9+16}=5$.

$\therefore$ the required equation of the circle is $(x-2)^2+(y-3)^2=25$,

or, $x^2+y^2-4x-6y-12=0$.

Ex. 2. Find the radius and centre of the circle

$$x^2+y^2-6x-8y-75=0.$$

The given equation may be written as $(x-3)^2+(y-4)^2=10^2$.

Hence the co-ordinates of the centre are (3, 4) and the radius is 10.

Ex. 3. Obtain the equation of the circle passing through the points (0, 0), (−1, 1) and (7, 7).

Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0. \quad \dots \quad \dots \quad \text{(i)}$$

Since the circle (i) passes through (0, 0), (−1, 1) and (7, 7),

$$\therefore \quad 0+0+0+0+c=0 \quad \dots \quad \dots \quad \text{(ii)},$$

$$2-2g+2f+c=0 \quad \dots \quad \dots \quad \text{(iii)},$$

and

$$98+14g+14f+c=0 \quad \dots \quad \dots \quad \text{(iv)}.$$

Solving (ii), (iii) and (iv), we have $g=-3$, $f=-4$ and $c=0$.

Substituting the values of g, f and c in (i), the required eqn. is

$$x^2+y^2-6x-8y=0.$$

Ex. 4. Prove that the straight line $y=2x+a\sqrt{5}$ touches the circle $x^2+y^2=a^2$.

By Art. 12, the straight line $y=mx+c$ touches $x^2+y^2=a^2$, if $c=a\sqrt{1+m^2}$.

Here $m=2$, $c=a\sqrt{5}$.

By the condition of the problem, $c=a\sqrt{1+4}=a\sqrt{5}$.

[The example can be worked out by solving the two equations.]

EXAMPLES III.

Find the equation to the circle :

1. Whose centre is (0, 0) and radius is 5.
2. Whose centre is (−3, 4) and radius is 6.
3. Whose centre is $(-a, -b)$ and radius is $(a+b)$.
4. Whose centre is $(0, -c)$ and radius is c.

Find the co-ordinates of the centres and the radii of the circles whose equations are the following :

5. $x^2+y^2-8x-10y-59=0$.
6. $x^2-10x+y^2=0$.
7. $x^2+y^2-2ux+2vy+d=0$.
8. $x^2+y^2+10y-39=0$.

Find the equations to the circles which pass through the points :

9. (6, 7), (−2, 7), (−2, −1).
10. (−3, 2), (−3, −10), (13, 2).
11. Show that the circle passing through (−3, 5), (4, 6), and (−2, −2) also passes through the point (6, 2).

12. Show that the points (a, b), $(-a, -b)$, $(-a, b)$, $(a, -b)$, (b, a), $(-b, -a)$, $(-b, a)$ and $(b, -a)$ are concyclic and find the equation of the circle through them.

13. Find the equation to the circle which touches each axis at a distance 8 from the origin.

14. Determine the equation to the circle which passes through the origin and cuts off intercepts equal to g and f from the positive parts of the axes.

15. Find the equation to the circle which passes through the origin and cuts off intercepts equal to 8 and 10 from the axes and determine its centre and radius.

16. Determine the radius of the circle which has its centre at $(-g, -f)$ and passes through the point (l, m) and find its equation.

17. Find the equation of the circle which has its centre at the point $(3, -10)$ and goes through the point $(11, -16)$.

18. Find the equation of the circle which has its centre at the origin and goes through the point of intersection of y-axis and $3x-5y-10=0$.

19. Find the equation of the circle which has its centre at the origin and meets the x-axis at the point where it is intersected by $\frac{x}{25}+\frac{y}{15}=1$.

20. Find the co-ordinates of the points of intersection of $\frac{x}{3}+\frac{y}{4}=1$ with the circle $x^2+y^2-3x-4y=0$.

21. Show that the straight line $y=mx+c\sqrt{1+m^2}$ is always a tangent to the circle $x^2+y^2=c^2$. Find the co-ordinates of the point of contact.

22. Find the co-ordinates of the points of intersection of $x^2+y^2=25$ with the straight line $x+3y-15=0$.

23. Find the equation of the circle passing through the origin and the points of intersection of $\frac{x}{4}+\frac{y}{6}=1$ with the axes of co-ordinates.

24. Show that the straight line $y-3x=10$ always touches the circle $x^2+y^2=10$.

CHAPTER IV.

THE PARABOLA.

14. *To find the equation to a parabola.*

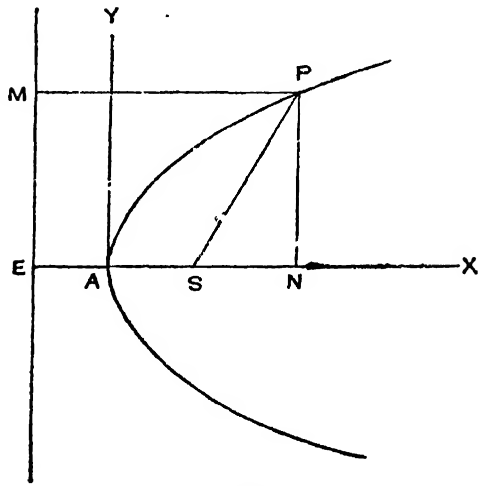

Fig. 7

Let AX, the axis of the parabola and AY. the tangent at the vertex of the parabola, be the axes of x, y respectively.

Let $P(x, y)$ be any point on the parabola with focus S and directrix ME.

Draw PN perpendicular to AX.

We know that in a parabola

$$PN^2 = 4AS.AN. \quad \ldots \quad \ldots \quad \text{(i)}$$

Now the vertex A being the origin,

$$AN = x,\quad PN = y \text{ and } AS = a \text{ (suppose)},$$

where the latus rectum $= 4AS = 4a$.

$\therefore$ from (i), $y^2 = 4ax$ is the equation to the parabola.

Note. The equation to the directrix ME is

$$x = -a, \text{ i.e., } x + a = 0, \text{ because } EA = AS = a.$$

Second Method : Equation deduced from the definition.

Join SP. Draw PM perpendicular on the directrix.

Then P being a point to the parabola, we have

$$SP = PM,$$

$$\therefore \quad SP^2 = PM^2 = EN^2 \quad \ldots \quad \text{(i)}$$

The co-ordinates of S being (a, o),

$$SP^2 = (x - a)^2 + y^2.$$

Again, $\quad EN = EA + AN = (a + x)$.

$\therefore$ from (i), we have $(x - a)^2 + y^2 = (a + x)^2$,

$$\text{i.e., } \mathbf{y^2 = 4ax}.$$

[The equation $y^2 = 4ax$ is the simplest form of the equation to a parabola.]

15. *To find the co-ordinates of the points of intersection of a given straight line and a parabola.*

Let the equations of the parabola and the straight line be $y^2 = 4ax$...(i) and $y = mx + c$...(ii), respectively.

In order to find the co-ordinates of the points of intersection, we have to solve (i) and (ii).

Substituting the value of y from (ii) in (i), we have

$$(mx + c)^2 = 4ax,$$

$$\text{i.e., } m^2x^2 + 2mxc + c^2 = 4ax,$$

$$\text{or, } m^2x^2 + 2x(mc - 2a) + c^2 = 0. \quad \ldots \quad \text{(iii)}$$

The roots of (iii) give the x-co-ordinates of the two points of intersection of (i) and (ii). When these roots are substituted in (ii), we get the y-co-ordinates of the two points of intersection of (i) and (ii). The two points of intersection are real or imaginary according as the roots of (iii) are real or imaginary.

The two points of intersection of (i) and (ii) will be coincident *i.e.*, the st. line $y = mx + c$ will be a tangent to the parabola $y^2 = 4ax$, if the roots of (iii) be real and equal,

$$\begin{aligned} &i.e., \text{ if } 4(mc-2a)^2 - 4m^2c^2 = 0, \\ &i.e., \text{ if } \quad (mc-2a)^2 - m^2c^2 = 0, \\ &\qquad\qquad \text{or, } \; a - mc = 0, \\ &\qquad\qquad\qquad \text{or, } \; c = \frac{a}{m}. \end{aligned}$$

Substituting the value of c in (ii), we find that the st. line $y = mx + \dfrac{a}{m}$ is always a tangent to the parabola $y^2 = 4ax$.

Ex. 1. Find the latus rectum and the co-ordinates of the focus of the parabola $y^2 = -4px$.

The equation $y^2 = -4px$ always represents a parabola passing through the vertex, whose concavity is turned towards the negative direction of the axis of x.

$$\text{Latus rectum} = 4AS = 4p.$$

The co-ordinates of the focus are $(-p, 0)$, because the distance of the focus from the vertex is one-fourth of the latus rectum.

Ex. 2. Find the length of the latus rectum of the parabola $5y^2 = 7x$, and also the co-ordinates of its focus. [C. U. 1936.]

The equation of the parabola may be written in the form $y^2 = \frac{7}{5}x$, so that the length of the latus rectum $= \frac{7}{5}$.

$\therefore$ one-fourth of the latus rectum or, $AS = \frac{7}{20}$.

Hence the co-ordinates of the focus are $(\frac{7}{20}, 0)$.

Ex. 3. Show that the straight line $y = 2x + 1$ touches the parabola $y^2 = 8x$, and find the co-ordinates of the point of contact.

Let us solve $y^2 = 8x$ ...(i) and $y = 2x + 1$ ... (ii).

Substituting the value of y in (i) from (ii), we have

$$(2x+1)^2=8x, \quad \text{or,} \quad 4x^2-4x+1=0,$$

$$\text{or,} \quad (2x-1)^2=0. \quad \therefore \; x=\tfrac{1}{2}, \tfrac{1}{2} \text{ and from (ii), } y=2, 2.$$

Hence the straight line cuts the parabola in two coincident points, *i.e.*, it touches the parabola and the point of contact is $(\frac{1}{2}, 2)$.

Ex. 4. Determine the co-ordinates of the points of intersection of the straight line $3y-x-8=0$ with the parabola $y^2=4x$.

From the equation of the straight line $3y-x-8=0$, $y=\dfrac{x+8}{3}$.

Substituting the value of y in $y^2=4x$, we have

$$\left(\frac{x+8}{3}\right)^2=4x, \quad \text{or,} \quad x^2-20x+64=0,$$

$$\text{or,} \quad (x-16)(x-4)=0. \quad \therefore \; x=16, 4.$$

Substituting the values of x in $3y-x-8=0$, $y=8, 4$.

Hence the co-ordinates of the points of intersection are $(16, 8)$ and $(4, 4)$.

Ex. 5. Show that the straight line $3y=5x+4$ touches the parabola $9y^2=80x$.

For the straight line $3y=5x+4$, $m=\frac{5}{3}$ and $c=\frac{4}{3}$. [*Art. 5.*]

For the parabola $9y^2=80x$, $4a=\frac{80}{9}$, whence $a=\frac{20}{9}$. [*Art. 13.*]

Hence, $\dfrac{a}{m}=\frac{20}{9}\times\frac{3}{5}=\frac{4}{3}=c$. [*Art. 14.*]

Hence the straight line touches the parabola.

Ex. 6. The parabola $y^2=4px$ goes through the point $(3, -2)$. Obtain the length of the latus rectum and the co-ordinates of the focus of this parabola. [C. U. 1934.]

The parabola $y^2=4px$ passes through $(3, -2)$.

$\therefore$ $4=4p\times 3$, whence $4p=\frac{4}{3}$ which is the length of the latus rectum $(=4AS)$.

The focus is always on the axis of the parabola, *i.e.*, the axis of x, and the distance of the focus from the vertex is equal to one-fourth of the latus rectum $(=AS)=\frac{1}{4}\times\frac{4}{3}=\frac{1}{3}$.

Hence the co-ordinates of the focus are $(\frac{1}{3}, 0)$.

EXAMPLES IV.

Find the *latera recta* and the co-ordinates of the foci of the following parabola :

1. $x^2=4by$. 2. $x^2=-4by$. 3. $3y^2=16x$. 4. $3y^2=-16x$.

5. Obtain the length of the latus rectum and the co-ordinates of the focus of the parabola $y^2 = 8px$, when it goes through the points (i) (4, 8) and (ii) (36, 12).

6. Show that the straight line $y = 2x + 3$ touches the parabola $y^2 = 24x$ and find the co-ordinates of its point of contact.

7. Determine the value of p, if the straight line $y = 3x + 5$ touches the parabola $y^2 = 8px$. Hence find the co-ordinates of the focus.

8. Taking the axis of the parabola and the tangent at the vertex as the axes of x and y respectively, find the equation of the parabola whose focus is the point (i) $(-5, 0)$; (ii) $(4, 0)$; (iii) $(0, -8)$; (iv) $(0, b)$.

9. Find the points of intersection of the straight line $5y - 4x - 24 = 0$ with the parabola $y^2 = 16x$.

10. Determine the points of intersection of the parabola

(a) $3y^2 = 5x$ with the straight line $2x = 3y$.

(b) $y^2 = 4ax$ with the straight line $2(x-a)a = (a+1)(y-2a)$.

11. A double ordinate of the parabola $y^2 = 4ax$ is of length $8a$. Show from geometrical considerations that the lines from the vertex to its two ends are at right angles.

12. Find the latus rectum and the co-ordinates of the focus of the parabola $3y^2 = 4x$, and determine the points in which it is met by the straight line $2x = 3y$. [C. U. 1935.]

[$y^2 = \frac{4}{3}x$. $\therefore$ latus rectum $= \frac{4}{3} = 4AS$, so that $AS = \frac{1}{3}$. The co-ordinates of the focus are $(\frac{1}{3}, 0)$. Solving the two equations $3y^2 = 4x$ and $2x = 3y$, the points of intersections are (0, 0) and (3, 2).]

13. Determine the co-ordinates of the focus of the parabola $x^2 = 4py$ which passes through the point of intersection of the straight lines $\frac{x}{2} + \frac{y}{5} = 1$ and $\frac{x}{5} + \frac{y}{2} = 1$.

14. Find the equations of the directrix, the axis and the latus rectum of the parabola $x^2 = 4ky$, and show that it is cut by the st. line $x = y + k$ in two coincident points.

15. Find the point of intersection of the lines $\frac{x}{3} + \frac{y}{2} = 1$ and $\frac{x}{2} + \frac{y}{3} = 1$, and determine the co-ordinates of the focus of the parabola $y^2 = 2ax$, which passes through this point. [C. U. 1943.]

CHAPTER V.

THE ELLIPSE.

16. *To find the equation to an ellipse.*

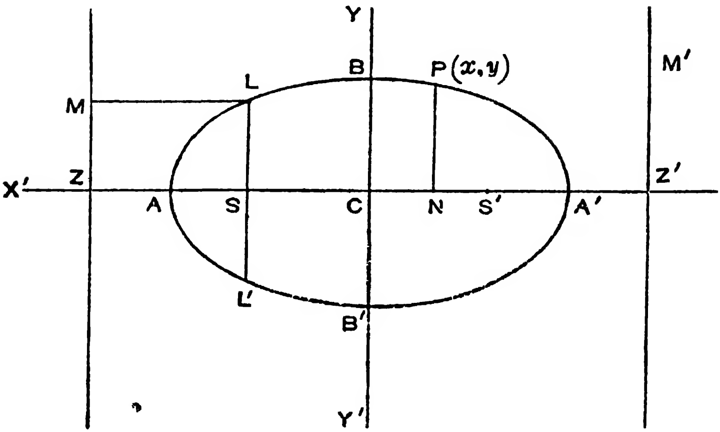

Fig. 8

Let AA' and BB' be the major and minor axes of the ellipse and C its centre. If P be any point on the ellipse and PN be drawn perpendicular upon the major axis, we know from geometry,

$$\frac{PN^2}{AN.A'N}=\frac{CB^2}{CA^2}.$$

Let us take C as the origin, and CA' and CB as the positive directions of the x and y axes respectively. If the co-ordinates of P be x, y, the relation subsisting between them gives us the equation to the ellipse.

Now, if $AA' = 2a$; $BB' = 2b$,

$$CA' = CA = a,\ CB' = CB = b,$$

[from the geometrical properties of the ellipse.]

From fig. 8, $AN = AC + CN = a + x$

$$A'N = A'C - CN = a - x$$

and $PN = y$.

Hence substituting the values in $\dfrac{PN^2}{AN.A'N} = \dfrac{CB^2}{CA^2}$,

we have $$\frac{y^2}{(a+x)(a-x)} = \frac{b^2}{a^2},$$

or, $a^2y^2 = b^2(a+x)(a-x) = b^2(a^2 - x^2)$,

$$\therefore\ b^2x^2 + a^2y^2 = a^2b^2.$$

Dividing both sides by a^2b^2, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

as the required equation of the ellipse.

Note. We know $CB^2 = CA^2(1-e^2)$, $\therefore$ $b^2 = a^2(1-e^2)$, where e is the eccentricity of the ellipse.

17. *To find the co-ordinates of the points of intersection of a given straight line and an ellipse.*

Let the equations of the straight line and the ellipse be $y = mx + c \cdots$(i) and $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \cdots$(ii), respectively.

In order to find the co-ordinates of the points of intersection, we have to solve the equations (i) and (ii).

Substituting the value of y from (i) in (ii), we have

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1,$$

i.e. $b^2x^2 + a^2(m^2x^2 + 2mxc + c^2)^2 = a^2b^2$,

or, $x^2(a^2m^2 + b^2) + 2mca^2x + a^2(c^2 - b^2) = 0. \quad \cdots$ (iii)

The two roots of (iii) give the x-co-ordinates of the two points of intersection of (i) and (ii). When these roots are substituted in (i), wet get the y-co-ordinates of the two points of intersection of (i) and (ii). The two points of intersection of the straight line and the ellipse may be real or imaginary according as the roots of (iii) are real or imaginary.

The two points of intersection of (i) and (ii) will be coincident *i.e.*, the straight line $y=mx+c$ will be a tangent to the ellipse 1 if the roots of (iii) be real and equal.

i.e., if $(2mca^2)^2-4(a^2m^2+b^2)a^2(c^2-b^2)=0$,

i.e., if $m^2c^2a^2-(a^2m^2+b^2)(c^2-b^2)=0$,

i.e., if $b^2a^2m^2+b^4-b^2c^2=0$,

i.e., if $a^2m^2+b^2-c^2=0$,

i.e., if $c^2=a^2m^2+b^2$.

Hence, $c=\pm\sqrt{a^2m^2+b^2}$.

Substituting the values of c in (i), we find that the straight lines $y=mx\pm\sqrt{a^2m^2+b^2}$ are always tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Note. We know that in an ellipse $CS=e.CA$. [*Prop. VI.*]

$\therefore\ e=\frac{CS}{CA}$. $\therefore\ e^2=\frac{CS^2}{CA^2}=\frac{a^2-b^2}{a^2}$

[$\because\ CS^2=SB^2-CB^2=a^2-b^2$.]

$=1-\frac{b^2}{a^2}$, whence $e=\sqrt{1-\frac{b^2}{a^2}}$.

We also know that in an ellipse $CA=e.CZ$. [*Prop. VI.*]

In fig, 8, $CZ=\frac{CA}{e}=\frac{a}{e}$.

Hence the equation of the directrix ZM is $x=-\frac{a}{e}$,

or, $x+\frac{a}{e}=0$, or, $ex+a=0$.

Similarly the equation of the directrix $Z'M'$ is $x = \frac{a}{e}$,

$$\text{or,} \quad x - \frac{a}{e} = 0, \qquad \text{or,} \quad ex - a = 0.$$

18. *To find the length of the latus rectum of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \cdots \qquad \cdots \text{(i)}$$

The latus-rectum is, by definition, the double ordinate through the focus.

The co-ordinates of a focus of (i) are $(ae, 0)$.

Hence in (i), putting $x = ae$, we have $y^2 = b^2(1 - e^2)$.

$$\therefore \quad y = b\sqrt{1 - e^2} = \frac{ba\sqrt{1 - e^2}}{a}.$$

$$\therefore \quad LSL' = 2y = 2\frac{ba\sqrt{1 - e^2}}{a} = 2\frac{b^2}{a}, \text{ from fig. 8,}$$

$$\because \quad b^2 = a^2(1 - e^2). \qquad \therefore \quad \textbf{Latus rectum} = 2\frac{b^2}{a}.$$

Ex. 1. Find the equation to the ellipse referred to its axes as axes of co-ordinates which passes through the points (2, 2) and (3, 1). Find also its eccentricity. [C. U. 1939.]

Let the equation to the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. $\cdots$ (i)

Since, it passes through (2, 2) and (3, 1),

$$\therefore \quad \frac{4}{a^2} + \frac{4}{b^2} = 1 \qquad \cdots \qquad \cdots \qquad \cdots \text{(ii)}$$

$$\text{and} \quad \frac{9}{a^2} + \frac{1}{b^2} = 1. \qquad \cdots \qquad \cdots \qquad \cdots \text{(iii)}$$

Solving (ii) and (iii), we have $a^2 = \frac{32}{3}$, $b^2 = \frac{32}{5}$.

Substituting the values of a^2, b^2 in (i), the required equation of the ellipse is $\frac{3x^2}{32} + \frac{5y^2}{32} = 1$.

The eccentricity $= \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{3}{5}} = \sqrt{\frac{2}{5}}$

Ex. 2. For what value of p does the ellipse $px^2+4y^2=1$ pass through the points $(\pm 1, 0)$? Find the length of its two axes.

[C. U. 1935.]

Substituting the co-ordinates of the points *i.e.*, $x=\pm 1$, $y=0$ in the equation $px^2+4y^2=1$, we have $p+0=1$, *i.e.*, $p=1$. The equation of the ellipse becomes $x^2+4y^2=1$,

$$\text{or,}\quad \frac{x^2}{1^2}+\frac{y^2}{(\frac{1}{2})^2}=1.$$

Hence the lengths of the semi-axes are 1 and $\frac{1}{2}$, so that the lengths of the axes are 2 and 1 respectively.

Ex. 3. Find the equation to the ellipse which meets the straight line $\frac{x}{7}+\frac{y}{2}=1$ on the axis of x and the straight line $\frac{x}{3}+\frac{y}{5}=1$ on the axis of y and whose axes lie along the axes of co-ordinates.

Determine the eccentricity and the positions of the foci of the ellipse.

[C. U. 1938.]

To find the point at which the straight line $\frac{x}{7}+\frac{y}{2}=1$ meets the x-axis, put $y=0$ in the equation $\frac{x}{7}+\frac{y}{2}=1$. $\therefore$ $\frac{x}{7}+0=1$, whence $x=7$, *i.e.* the straight line meets the x-axis at the point (7, 0).

Similarly putting $x=0$ in the equation $\frac{x}{3}+\frac{y}{5}=1$, we have $y=5$. Hence the straight line $\frac{x}{3}+\frac{y}{5}=1$ meets the y-axis at the point (0, 5).

But the axes of the ellipse lie along the axes of co-ordinates.

$\therefore$ the semi-axes of the ellipse are 7 and 5.

Hence the equation of the ellipse is $\frac{x^2}{7^2}+\frac{y^2}{5^2}=1$, or, $\frac{x^2}{49}+\frac{y^2}{25}=1$.

The eccentricity of the ellipse $=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{25}{49}}$

$$=\sqrt{\frac{24}{49}}=\frac{2}{7}\sqrt{6}.$$

The co-ordinates of the foci are (ae, o) and $(-ae, o)$, because

$$CS=CA.e=ae.$$

$$\text{But}\quad a.e=7\times\frac{2}{7}\sqrt{6}=2\sqrt{6}.$$

Hence the required co-ordinates of the foci, S'. S are $(2\sqrt{6}, 0)$ and $(-2\sqrt{6}, 0)$ respectively. [*See fig. 8.*]

Ex. 4. Find the eccentricity and the positions of the foci of the ellipse $x^2+2y^2=2$. [C. U. 1937.]

The equation of the ellipse may be written in the form

$$\frac{x^2}{2}+\frac{y^2}{1}=1, \text{ so that, } a^2=2,\ b^2=1.$$

$$\therefore\ a=\sqrt{2} \text{ and } b=1. \quad \therefore\ e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{1}{2}}=\frac{1}{2}\sqrt{}.$$

The co-ordinates of the foci are (ae, o) and $(-ae, o)$,

$$\text{but } a.e=\sqrt{2}\times\frac{1}{\sqrt{2}}=1.$$

Thus the required co-ordinates of the foci are $(1, 0)$ and $(-1, 0)$, which determine their positions.

EXAMPLES.

1. Find the equation of the ellipse whose latus rectum is $4\frac{1}{2}$ and eccentricity is $\sqrt{\frac{7}{4}}$, if the centre of the ellipse be the origin and the axes be the axes of co-ordinates.

[Latus rectum $=\dfrac{2b^2}{a}=4\frac{1}{2}$......(i). [*Art. 17.*]

Again, $e^2=\dfrac{a^2-b^2}{a^2}=\dfrac{7}{16}$......(ii)

Solving (i) and (ii), $a^2=16$, $b^2=9$.]

2. Determine the equations of the ellipses, whose centres are coincident with the origin, whose axes are the axes of co-ordinates, if they pass through (i) the points $\left(\frac{5}{\sqrt{2}}, 2\sqrt{2}\right)$ and $(5, 0)$,

(ii) the points $(1, 4)$ and $(-6, 1)$,

and (iii) the points $(3, 4)$ and $(2, 2\sqrt{5})$.

3. If the two foci of an ellipse coincide, show that the ellipse becomes a circle.

[If the foci coincide, they must be coincident with the centre C of the ellipse, because $CS=CS'=a.e.$

$\therefore\ SS'=2CS$ being zero, $2a.e=0$, whence $e=0$.

$\therefore\ e^2=\dfrac{a^2-b^2}{a^2}=0$, whence $a^2=b^2$. Hence the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is reduced to the circle $x^2+y^2=a^2$.]

4. Find the equation of the ellipse referred to its centre and axes, whose foci are the points (5, 0) and (−5, 0) and whose eccentricity is $\frac{1}{3}$.

5. Show that the straight line $x+3y=5$ touches the ellipse $4x^2+9y^2=20$ and find its point of contact.

6. Find the co-ordinates of the point at which the straight line $x=4$ is a tangent to the ellipse $9x^2+16y^2=144$.

7. Prove that the straight line $x+2y-8=0$ touches the ellipse $3x^2+4y^2=48$, and determine its point of contact.

8. Find the latus rectum, the eccentricity and the co-ordinates of the foci of the following ellipses :

(i) $\frac{x^2}{25}+\frac{y^2}{16}=1.$ (ii) $9x^2+162=144.$

9. Find the value of p for which the ellipse $4x^2+py^2=80$ passes through the points (0, ± 4), and find its eccentricity.

10. Find the points of intersection of the straight line $5x+2y-30=0$ with the ellipse $\frac{x^2}{4}+\frac{y^2}{5}=9$.

11. Determine the points of intersection of the line $\frac{x}{p}+\frac{y}{q}=1$ with the ellipse $\frac{x^2}{p^2}+\frac{y^2}{q^2}=1$.

12. Find the equation of the ellipse referred to its centre as the origin and its axes as the axes of x and y respectively, which passes through the points at which the straight line $\frac{x}{5}+\frac{y}{4}=1$ meets the co-ordinate axes.

MISCELLANEOUS EXAMPLES.

1. A straight line forms a right-angled triangle with the axes of co-ordinates. If the hypotenuse is 13 and the area of the triangle is 30, find the equation of the straight line. [C. U. 1938.]

[Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$. Then by hypothesis, $a^2+b^2=13^2=169$ ... (i) and $\frac{1}{2}ab=30$, or, $ab=60$ ... (ii).

Solving (i) and (ii), $a=12, -12, 5, -5$; $b=5, -5, 12, -12$.

Hence the four straight lines are $\frac{x}{12}+\frac{y}{5}=1$, $\frac{x}{12}+\frac{y}{5}=-1$,

$$\frac{x}{5}+\frac{y}{12}=1 \text{ and } \frac{x}{5}+\frac{y}{12}=-1.\ ]$$

2. Find the equation of the ellipse which meets the straight line $\frac{x}{16}+\frac{y}{19}=1$ on the axis of x and the straight line $\frac{x}{21}+\frac{y}{2}=1$ on the axis of y and whose axes are the axes of co-ordinates.

3. For what point on the parabola $y^2=8x$ is the ordinate equal to four times the abscissa ?

4. Prove that the parabola is reduced to a pair of coincident straight lines, if the distance between the focus and the vertex is reduced to zero. [$y^2=4ax=4\times0\times x=0$.]

5. If the semi-minor axis be equal to the distance of the focus from the centre, find the eccentricity.

6. Determine the value of p, if the parabola $y^2=4px$ passes through the point (16, 8). Hence show that the parabola also passes through the points (4, 4) and (9, 6).

7. Find the co-ordinates of the points of intersection of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{\beta^2}=1$ with the axes of co-ordinates. Show that the quadrilateral formed by the four points of intersection is a parallelogram. Determine the magnitude of its area.

8. Show that the ellipse reduces to a pair of coincident straight lines, if the two foci coincide with the respective vertices.

[$ae=a$, whence $e=1$. $\therefore$ $b^2=a^2(1-e^2)=0$.]

9. A straight line forms a right-angled triangle of area 6 with the axes of co-ordinates and intercepts a distance of 5 between them. Find its equation.

10. Find the equation of the circle which touches the axes at (0, 1) and (1, 0). [C. U. B.A. & B.Sc., 1926, '31.]

11. Show that the circle $x^2+y^2-2ax-2ay+a^2=0$ touches the axes of x and y, and find the chord of contact.

[Putting $x=0$, which is the equation to y-axis, we have $(y-a)^2=0$. $\therefore$ the circle touches y-axis at $(0, a)$. Similarly the circle touches the x-axis at $(a, 0)$. The chord of contact is the st. line joining the pts. of contact *i.e.* $(0, a)$ and $(a, 0)$.]

12. Find the areas of the triangles cut off from the axes of co-ordinates by (i) $\frac{x}{14}+\frac{y}{5}=1$, and (ii) $3x+4y=60$.

13. Find the point of intersection of the lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$. [C. U. B.A. & B.Sc., 1928.]

14. Find the (i) axes, (ii) eccentricity and (iii) foci of the ellipse $p^2x^2+q^2y^2=1$.

15. Find the equation of the circle which touches the y-axis at a distance 5 from the origin and whose centre is at a distance 3 from the y-axis.

16. Show that the ellipse $\frac{x^2}{4^2}+\frac{y^2}{3^2}=1$ passes through the points $(4, 0)$, $(0, 3)$, and $\left(\pm\frac{4\sqrt{2}}{\sqrt{3}}, \pm\sqrt{3}\right)$.

17. Show that the three straight lines $2x-y-1=0$, $y-x-1=0$ and $x+y-5=0$ meet in a point. Find also the equation of the line joining the common point of intersection of the three lines with the origin.

18. Show that the five points $(8, 0)$, $(0, 6)$, $(4, 3)$, $(12, -3)$, and $(-8, 12)$ lie on a straight line and find the equation of the straight line passing through them.

19. Prove that the straight lines $y=\pm(x+2)$ touch the parabola $y^2=8x$ at the ends of the latus rectum.

20. Find the co-ordinates of the point of intersection of the straight line $4y-x=24$ with the parabola $y^2=6x$.

21. Show that the straight line $y-3x=10$ cuts the circle $x^2+y^2=10$ in two coincident points, and determine the co-ordinates of this point. [*See Examples III, Ex. 24.*] [C. U. 1943.]

22. The distance between the focus and the directrix of an ellipse is 16 inches and its eccentricity is $\frac{3}{5}$. Obtain the lengths of the principal axes.

[From fig. 8, $CZ=\frac{a}{e}$, $CS=ae$. $\therefore$ $SZ=CZ-CS=\frac{a}{e}-ae$.

But $e=\frac{3}{5}$, $\therefore$ $a=15$ inches. [*Art. 16.*]

Again, $b^2=a^2(1-e^2)$, whence $b=12$ inches. (*Art. 17*)]

ANSWERS

Examples I.

1. (i) a ; (ii) $\sqrt{a^2+b^2+c^2+d^2-2ac-2bd}$; (iii) 5 ; (iv) $2a \sin\left(\frac{\theta-\phi}{2}\right)$; (v) 10 ; (vi) $5\sqrt{13}$. 2. 8.
3. 5, 15. 5. 11, 2.

Examples II.

1. $\frac{x}{3}-\frac{y}{3}=1$. 2. $\frac{y}{8}-\frac{x}{7}=1$. 3. $\frac{x}{5}+\frac{y}{4}+1=0$. 4. $x-y=2$.
5. $5x-12y=60$. 6. $12x-7y=3$. 7. $\frac{x}{a}+\frac{y}{b}=1$. 8. $cy=dx$.
9. $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$. 10. $y-x-1=0$; $\sqrt{2}$.
11. $\left(\frac{c}{a}, 0\right)$, $\left(0, \frac{c}{b}\right)$; $\frac{c\sqrt{a^2+b^2}}{ab}$. 12. (i) (3, 4), (ii) $(1\frac{1}{5}, 1\frac{1}{5})$.
13. (7, 5). 16. 1. 18. $2x-y-7=0$. 20. $4x-3y-31=0$.

Examples III.

1. $x^2+y^2=25$. 2. $x^2+y^2+6x-8y-11=0$.
3. $x^2+y^2+2ax+2by-2ab=0$. 4. $x^2+y^2+2cy=0$.
5. (4, 5), 10. 6. (5,0), 5. 7. $(u, -v)$, $\sqrt{u^2+v^2-d}$.
8. $(0, -5)$, 8. 9. $x^2+y^2-4x-6y-19=0$.
10. $x^2+y^2-10x+8y-59=0$. 11. $x^2+y^2-2x-4y-20=0$.
12. $x^2+y^2=a^2+b^2$. 13. $(x-8)^2+(y-8)^2=64$.
14. $x^2+y^2-gx-fy=0$. 15. $x^2+y^2-8x-10y=0$.
16. $\sqrt{(l+g)^2+(m+f)^2}$, $(x+g)^2+(y+f)^2=(l+g)^2+(m+f)^2$.
17. $(x-3)^2+(y+10)^2=100$. 18. $x^2+y^2=4$. 19. $x^2+y^2=625$.
20. (3, 0), (0, 4). 21. $\left(\frac{-mc}{\sqrt{1+m^2}}, \frac{c}{\sqrt{1+m^2}}\right)$.
22. (3, 4), (0, 5). 23. $x^2+y^2-4x-6y=0$.

Examples IV.

1. $4b$, $(0, b)$. 2. $4b$, $(0, -b)$. 3. $\frac{16}{3}$, $(\frac{4}{3}, 0)$.

4. $16 \left(\frac{-4}{3}, 0\right)$. **5.** (i) $16, (4, 0)$; (ii) $4, (1, 0)$.

6. $\frac{3}{2}, 6$. **7.** $\frac{15}{2}, (15, 0)$. **8.** (i) $y^2 = -20x$; (ii) $y^2 = 16x$; (iii) $x^2 = -32y$; (iv) $x^2 = 4by$.

9. $(4, 8), (9, 12)$. **10.** (*a*) $(0, 0), \left(\frac{15}{4}, \frac{5}{2}\right)$; (*b*) $(a, 2a), \left(\frac{1}{a}, 2\right)$.

13. $(0, \frac{5}{14})$. **14.** $y+k=0$; $x=0$; $y=k$. **15.** $(1\frac{1}{5}, 1\frac{1}{5})$; $(\frac{3}{10}, 0)$.

Examples V.

1. $\frac{x^2}{16}+\frac{y^2}{9}=1$. **2.** (i) $\frac{x^2}{25}+\frac{y^2}{16}=1$. (ii) $3x^2+7y^2=115$.

(iii) $4x^2+5y^2=116$. **4.** $\frac{x^2}{625}+\frac{y^2}{600}=1$. **5.** $\left(1, \frac{1}{3}\right)$.

6. $(4, 0)$. **7.** $(2, 3)$. **8.** (ii) $6\frac{2}{5}$; $\frac{3}{5}$; $(3, 0), (-3, 0)$.

(ii) $4\frac{1}{2}$; $\frac{\sqrt{7}}{4}$; $(\sqrt{7}, 0), (-\sqrt{7}, 0)$. **9.** $5, \frac{1}{\sqrt{5}}$.

10. $(4, 5), (6, 0)$. **11.** $(p, 0), (0, q)$. **12.** $\frac{x^2}{25}+\frac{y^2}{16}=1$.

Miscellaneous Examples.

2. $\frac{x^2}{256}+\frac{y^2}{81}=1$. **3.** $\left(\frac{1}{2}, 2\right)$. **5.** $\frac{1}{\sqrt{2}}$. **6.** 1.

7. $(a, 0), (0, \beta), (-a, 0), (0, -\beta)$; $2a\beta$.

9. $\frac{x}{4}+\frac{y}{3}=\pm 1$, $\frac{x}{3}+\frac{y}{4}=\pm 1$. **10.** $x^2+y^2-2x-2y+1=0$.

11. $x+y=a$. **12.** (i) 35 ; (ii) 150. **13.** $x=y=\frac{ab}{a+b}$.

14. (i) $\frac{2}{p}$; $\frac{2}{q}$; (ii) $\frac{\sqrt{q^2-p^2}}{q}$; (iii) $\pm\frac{\sqrt{q^2-p^2}}{pq}$, 0.

15. $x^2+y^2-6x-10y+25=0$. **17.** $3x-2y=0$. **18.** $3x+4y=24$.

20. $(24, 12)$. **21.** $(-3, 1)$.

SOLID GEOMETRY

LINES AND PLANES.

CHAPTER I.

Definitions. 1. A **point** has only position, but no dimension,* that is, it has neither length, nor breadth, nor thickness.

2. A **line** has length, but it has neither breadth nor thickness, *i.e.*, it has one dimension.

3. A **surface** has length and breadth, but it has no thickness, *i.e.*, it has two dimensions.

4. A **solid** has length, breadth and thickness, *i.e.*, it has three dimensions.

5. A **plane** or a **plane surface** is a surface such that the straight line joining any pair of points in it lies wholly in the surface.

Note. A line is generated by the motion of a point. Lines therefore meet in points. A surface may be generated by the motion of a line. Surfaces meet in lines. A solid may be generated by the motion of a surface, so that a solid is bounded by surfaces. A line intersects a surface in points.

6. Solid Geometry deals with the properties of solids, planes, lines and points in a three dimensional space.

7. Lines or points which are in the same plane, or through which a plane may be made to pass are said to be **coplanar**.

*The length, breadth and thickness are each called a **dimension** of a body.

8. Two straight lines are said to be **parallel** when being in the same plane they do not meet though indefinitely produced.

9. Two straight lines through which a plane cannot be made to pass are said to be **skew** or *non-coplanar.*

Note. Skew lines never meet, however far they may be produced ; on the other hand they are not parallel, because they are non-coplanar.

The angle between two skew straight lines is the angle between one of them and the straight line drawn through any point in that line parallel to the other.

10. A straight line is said to be *parallel to a plane,* when it does not meet the plane though indefinitely produced.

11. Two or more **planes** are said to be **parallel** when they do not meet though they are indefinitely produced.

Note. It should be noted that in Solid Geometry straight lines are supposed to be of infinite length, planes of infinite extent, unless otherwise stated.

12. A straight line is said to be **perpendicular** or **normal to a plane,** when it is perpendicular to every straight line that meets it in that plane.

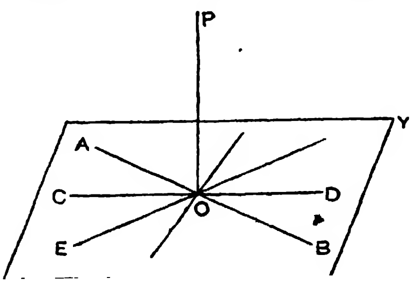

Fig. 1.

The straight line PO is perpendicular to the plane XY, when it is perpendicular to all lines like AOB, COD, EO etc. lying in the plane XY and meeting PO at O, the pt. where PO meets the plane XY.

13. A straight line or a plane is said to be **vertical,** when the straight line or the plane is parallel to the direction of a plumb line hanging freely at rest.

14. A plane is said to be **horizontal,** when it is perpendicular to a vertical line.

15. A straight line is said to be **horizontal,** when it lies in a horizontal plane.

Axioms.

1. A straight line joining any two points in a plane must lie wholly in the plane even if it be produced indefinitely.

2. An infinite number of planes may be made to pass through any two points or through any straight line.

Note. **If a plane passing through any given straight line or the straight line joining the two given points, be rotated about it, it will pass through an infinite number of positions.**

3. When a plane of infinite extent is rotated about any fixed straight line lying in it, it may be made to pass through any point in space outside the given line.

Note 1. (1) If a straight line be parallel to a plane, there is no common point between them. (2) A straight line which intersects a plane has only one point in common with the plane. (3) But a straight line which lies wholly in a plane has all the points (*i.e.* innumerable points) in common with the plane. Thus a straight line may be related to a plane in three ways.

Note 2. If two straight lines be coplanar, they must either intersect or be parallel.

If two straight lines be non-coplanar, they must neither intersect nor be parallel, *i.e.*, they are skew.

16. A quadrilateral which has not all its sides in one plane is called a **skew** or **gauche quadrilateral.**

Two adjacent sides of a skew quadrilateral lie in one plane and the other two adjacent sides lie in a different plane. If a plane quadrilateral be partly folded about one of its diagonals, we get a skew quadrilateral.

PROPOSITION I.

One, and only one plane may be made to pass through any two intersecting straight lines.†

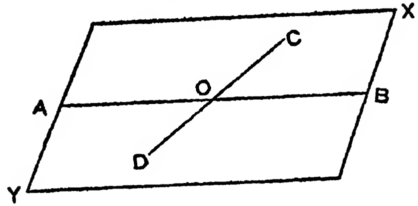

Fig. 2.

Let AB and CD be two straight lines intersecting at O.

It is required to prove that one and only one plane may be made to pass through AB and CD.

Take any plane passing through the line AOB and rotate this plane about the line AB until it passes through C and let the plane in its new position be denoted by XY. The plane now becomes fixed in position.

Since the points C and O lie on the plane XY,

the straight line CO or COD lies wholly in it.

But AB lies in XY by assumption ; hence the plane XY passes through both AB and CD.

In any other position of the rotating plane, the point C falls outside it, so that the straight line CD can not lie in it.

† C. U. 1912

Hence one, and only one plane can pass through AB and CD.

Note *1. One, and only one plane passes through a straight line and a point outside it.

Note 2. From all that has been said before it becomes obvious that the position of a plane is fixed, if it passes through—

(*a*) any straight line and a point outside it ;

(*b*) any two parallel straight lines ;

(*c*) any two intersecting straight lines ;

(*d*) any three points which are not collinear.

EXERCISES.

1. Any number of mutually intersecting straight lines, not passing through the same point, are coplanar.

2. *Any three straight lines forming a triangle are coplanar.* [C. U. 1911.]

3. One, and only one plane may be made to pass through a pair of parallel straight lines.

4. *Show that if three or more parallel straight lines intersect a given straight line, they are coplanar.* [C. U. 1915, '21.]

5. *Through a given point draw a straight line which intersects two given straight lines not lying in one and the same plane.*

[Through the given pt. draw a st. line parallel to one of the skew lines and produce the plane of parallels to meet the other skew line.]

6. *Draw a straight line to cut three given skew or non-intersecting straight lines.* [C. U. 1913.]

[AB, CD, EF are skew lines. Through P in CD draw a st. line $\parallel$ to a skew line. Produce the plane of parallels to meet the other skew line.]

7. If three or more concurrent straight lines cut a given straight line, they are coplanar.

8. If the diagonals of a quadrilateral intersect, the sides of the quadrilateral are coplanar with the diagonals.

9. *Straight lines joining the extremities of two skew straight lines are themselves skew st. lines.* [Four lines form a skew quadrilateral.]

*The truth of the statement is so fundamental that it can be regarded as an axiom.

Proposition II.

*Two intersecting planes cut one another in a straight line and in no point outside it.**

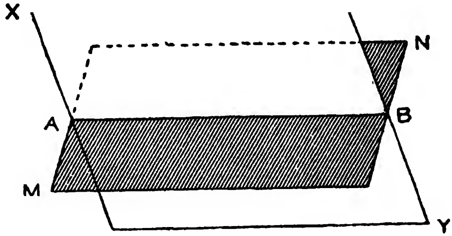

Fig. 3.

Let XY and MN be two intersecting planes.

It is required to prove that the two intersecting planes cut in a st. line, and in no point outside it.

Let A and B be any two points common to both the planes.

Since the points A and B lie on both the planes, the st. line AB, joining them, lies wholly in both the planes, so that the planes XY and MN intersect along the st. line AB.

Again, since both the planes pass through the st. line AB, they can have no point common to them outside AB, because in that case the two planes become one which is contrary to the hypothesis.

Cor. If two planes have one point in common, they will have an infinite number of common points.

Note. (1) A plane may be generated by a st. line which moves parallel to itself and slides over a fixed straight line.

(2) A plane may be generated by a st. line which slides over two fixed intersecting or parallel st. lines.

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* C. U. 1911, '13, '15, '21, '24, '41, '43.

(3) A plane may be generated by a st. line rotating about a fixed point and sliding over a fixed st. line.

Thus there are three ways of generating a plane.

EXERCISES.

1. *Prove that the common section of any three planes (non-collinear) meet at a point.* [C. U. 1911.]

2. The lines of intersection of two parallel planes with any third plane are parallel.

3. Two intersecting straight lines can not both be parallel to a third straight line.

PROPOSITION III.*

If a straight line perpendicular to each of two intersecting straight lines at their point of intersection, it is perpendicular to the plane in which they lie.

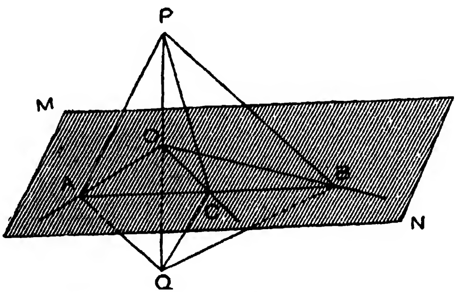


Fig. 4.

Let OP be perp. to each of the intersecting st. lines OA, OB at their point of intersection O.

As OA and OB are two intersecting st. lines they determine a plane ; let it be MN.

* C. U. 1923, '25, '30, '37, '39, '41.

It is required to prove that OP is perp. to the plane MN which contains OA, OB.

Through the point O, draw any st. line OC in the plane MN. It will be enough to shew that OP is perp. to OC. Draw the st. line AB cutting OA, OC, OB at the points A, C and B respectively.

Produce PO to Q, so that OQ is equal to OP.

Join PA, PC, PB and QA, QC, QB.

OA is perp. to PQ by hypothesis and O is the middle point of PQ.

$\therefore$ A is equidistant from P and Q. That is, $AP = AQ$.

Similarly OB being the perpendicular bisector of PQ, $BP = BQ$.

Now in the Δs APB, AQB,

$AP = AQ$, AB is common, and $BP = BQ$,

$\therefore$ the triangles are congruent, so that

the $\angle PAB =$ the $\angle QAB$. ... (i)

Again in the Δs RAC, QAC,

$AP = AQ$, AC is common, and

from (i), the $\angle PAC =$ the $\angle QAC$.

$\therefore$ the triangles are congruent, whence $PC = QC$ ... (ii)

Hence in the Δs POC, QOC,

$OP = OQ$, OC is common, and from (ii), $PC = QC$.

$\therefore$ the Δs are congruent, so that the $\angle POC =$ the $\angle QOC$; but these are supplementary angles; therefore each of them is a right angle, *i.e.*, PO is perp. to OC.

But OC is *any* st. line in the plane MN meeting OP at O.

$\therefore$ OP is perp. to the plane MN, which contains OA, OB.

Alternative proof :

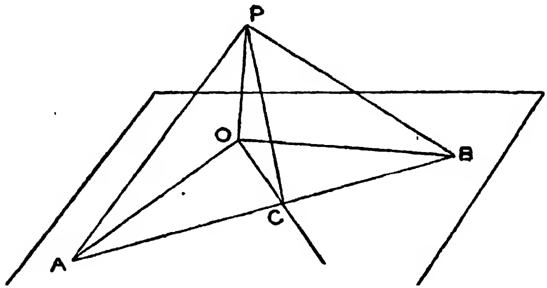


Fig. 4(a).

Through C, any point on OC, draw a line AB cutting OA and OB at A and B respectively and being bisected at the point C*.

Join PA, PB and PC.

As OP is perpendicular to OA, we have

$$\left.\begin{aligned} PA^2 &= OA^2 + OP^2 \\ \text{Similarly} \quad PB^2 &= OB^2 + OP^2 \end{aligned}\right\} \quad \cdots \quad (1)$$

But in the $\triangle PAB$, PC is a median,

$$\therefore \quad PA^2 + PB^2 = 2(PC^2 + AC^2). \quad \cdots \quad (2)$$

Similarly from the $\triangle OAB$,

$$OA^2 + OB^2 = 2(AC^2 + OC^2), \quad \cdots \quad (3)$$

i.e., from (2) and (3), $(PA^2 - OA^2) + (PB^2 - OB^2)$
$$= 2(PC^2 - OC^2),$$

i.e., $OP^2 + OP^2 = 2OP^2 = 2(PC^2 - OC^2)$, from (1),

i.e., $OP^2 = PC^2 - OC^2$,

or, $PC^2 = OP^2 + OC^2$ *i.e.*, OP is perpendicular to OC.

---

* Produce OC to D, so that $OC = OD$. Through D, draw lines parallel to OA and OB forming the parallelogram $OADB$. Then the diagonal AB is bisected at C.

EXERCISES.

1. Find the locus of a point in space equidistant from two given points. [C. U. 1915, '23, '39.]

[The plane bisecting the line joining the given points at right angles.]

2. All points in space equidistant from three non-collinear points lie in a st. line : [*viz.*, the intersection of two planes.] [C. U. 1941.]

3. Show that all points on the circumference of a circle are equidistant from any point on the normal to the plane of the circle passing through its centre. [C. U. 1937.]

4. Show that there is only one point equidistant from four non-coplanar points.

5. From any point O three straight lines OX, OY and OZ are drawn, so that each is perpendicular to the plane passing through the other two. Prove that they are mutually at right angles. Give a common illustration of the same.

6. How many straight lines may be drawn perpendicular to a given straight line at a given point ?

7. If two isosceles triangles, not in the same, plane have a common base, show that the common base will be perpendicular to the plane passing through its middle point and the vertices of the triangles.

8. A point which is not in the plane of a right-angled triangle is equidistant from the angular points. Show that the line joining it to the middle point of the hypotenuse is perpendicular to the plane of the triangle.

Proposition IV.*

All straight lines drawn perpendicular to a given straight line at a given point are coplanar.

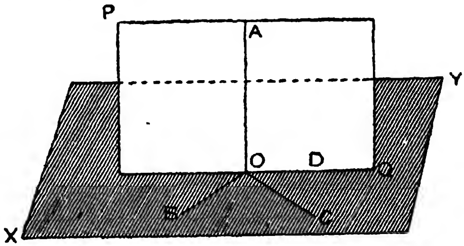


Fig. 5.

Let the st. lines OB, OC, OD be perpendicular to AO at the point O.

It is required to prove that OB, OC, OD are coplanar i.e., lie in one plane.

Let XY be the plane passing through OB, OC and let PQ be the plane passing OA, OD and intersecting the plane XY in the st. line OQ.

Since the st. lines OB, OC are perpendicular to OA.

$\therefore$ OA is perpendicular to the plane XY.

$\therefore$ the line OQ, which is in the plane of OB, OC, is perpendicular to OA. [*Prop. III.*]

Hence OQ and OD are both perpendicular to the same st. line OA and they are in the same plane PQ.

$\therefore$ OQ coincides with OD.

Hence OB, OC, OD are in the same plane XY.

Cor. If a right angle rotates about one of its sides containing the right angle, the other side generates a plane. [C. U. 1919.]

*C. U. 1916, '27, '32, '33, '36, '42.

EXERCISES.

1. How many horizontal lines can be drawn through a given point in a vertical line and how do they lie? [C. U. 1916.]

2. Show that the four corners of a horizontal rectangle are equidistant from any point on the vertical line passing through the centre of the rectangle.

3. If a triangle revolves about its base, show that the vertex describe a circle. [C. U. 1919.]

4. Prove that a point can be found in a plane equidistant from three points outside the plane. State the exceptional case if any.
[C. U. 1932, '35.]
[*See Prop. III, Ex. 2.*]

5. In any number of planes have a common line of section, show that the normals to these planes from any point on that line are coplanar.

6. Prove that there cannot be more than three mutually perpendicular st. lines in space meeting at a point. [C. U. 1932, '36.]

7. Find a point in a given st. line in space equidistant from two given points outside the st. line. Is there any exceptional case?
[*Prop. III, Ex. 1.*]

8. Prove that all straight lines drawn perpendicular from a given point to a system of parallel st. lines in space are coplanar.
[C. U. 1927.]

9. How many vertical lines can pass through a given point?

Proposition V.

*If two straight lines are parallel, and if one of them is perpendicular to a plane, the other also is perpendicular to the same plane.**

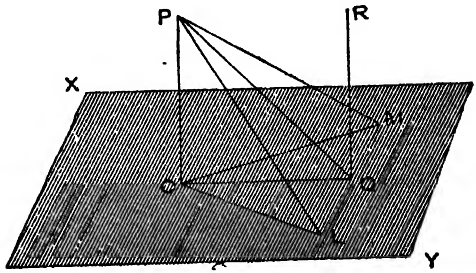


Fig. 6.

Let OP, QR be two parallel st. lines meeting the plane XY at O and Q respectively and let OP be perpendicular to the plane.

It is required to prove that QR also is perpendicular to the plane XY.

Join OQ, PQ. Through Q draw LQM in the plane XY perpendicular to OQ, such that $LQ = QM$.

Join OL, OM and PL, PM.

Since OQ is the perpendicular bisector of LM,

$$\therefore \quad OL = OM.$$

Now in the Δs POL, POM, we have $OL = OM$, OP common, and the $\angle POL =$ the $\angle POM$, each being a right angle, because OP is perpendicular to the plane of OL, OM.

Hence the Δs are congruent, so that $PL = PM$.

*C. U. 1910, '18, '28, '40, '42.

Again, in the Δs PLQ, PMQ, we have
$PL = PM$, PQ common and $LQ = QM$.
$\therefore$ the Δs are congruent.

$\therefore$ the $\angle PQL =$ the $\angle PQM$, whence PQ is perp. to LM.

But by construction, OQ is perp. to LM.
$\therefore$ LM is perp. to the plane of OQ, PQ.

But the plane of the parallels OP, QR, also contains OQ and PQ.

$\therefore$ LQM is perp. to the plane of the parallels, so that LQ is perp. to QR.

Again, OP, QR are parallel and OQ meets them.
$\therefore$ the $\angle POQ =$ the $\angle RQO =$ a right angle.

Hence RQ is perp. to QL as well as QO,

i.e., RQ is perp. to the plane XY which contains them.

Converse Proposition :

If two straight lines are perpendicular to the same plane, they are parallel to one another. [C. U. 1928.]

Let the st. lines OP, QR be both perpendicular to the plane XY meeting it at O and Q respectively.

It is required to prove that OP and OQ are parallel to one another.

Join OQ, PQ, as before, and through Q draw LM in the plane XY perp. to OQ, such that $LQ = QM$.
Join OL, OM and PL, PM.

Since OQ is the perp. bisector of LM,
$\therefore$ $OL = OM$.

The Δs OLP, OMP are congruent as before,
so that $PL = PM$.

Again, the Δs PLQ, PMQ are congruent as before,

so that the $\angle PQL$ = the $\angle PQM$,

whence PQ is perp. to LM.

But by construction, OQ is perp. to LM.

Hence LQ is perp. to the plane of QO, QP.

But by hypothesis, LQ is perp. to QR, so that QR is in the plane of QO and QP. [Prop. IV.]

But OP is also in the plane QO, QP.

$\therefore$ OP and QR are coplanar.

Again, each of the $\angle s$ POQ, RQO is a right angle.

[Definition of $\perp^r$ to a plane.]

$\therefore$ the st. lines OP, QR are parallel.

Cor. **If OP is perpendicular to any plane XY, and if from O, the foot of the perpendicular OP, a straight line OQ is drawn perpendicular to any straight line LM in the same plane, then PQ is also perpendicular to LM.** [C. U. 1938, '40.]

[Make $LQ = QM$ as before. Join OL, OM and PL, PM. The proof is exactly the same as in Prop. V.]

This theorem is known as "The Theorem of the Three Perpendiculars."

EXERCISES.

1. Draw a straight line perpendicular to a given plane XY from an external point P. [From P draw PQ perpendicular to LM. Draw QO perp. to LM in the plane XY. The perp. PO drawn from P on OQ is perp. to the plane XY.] [*See fig. 6.*]

2. Two perpendiculars are drawn on two intersecting planes from a point outside the planes. Show that the common line of intersection of the planes will be at right angles to the plane in which the two perpendiculars lie.

3. Straight lines in space which are parallel to a given straight line are parallel to one another.

[C. U. 1909, '14, '19, '22, '25, '29, '35.]

[Let the st. lines EF, GH be parallel to the given st. line PQ. *It is required to prove that EF, GH are parallel to one another.* Draw

any plane XY perp. to PQ. Since EF is parallel to PQ. $\therefore$ EF is perp. to the plane XY. Similarly GH is perp. to XY. Hence, EF, GH are both parallel to one another.]

4. If the middle points of adjacent sides of a skew quadrilateral are joined, prove that the figure so formed is a parallelogram.

5. If PQ, LM, RS be three equal parallel and non-coplanar straight lines, show that the triangles PLR, QMS are congruent.

6. If perpendiculars are drawn from any point to a system of parallel st. lines in space, then all the perpendiculars lie in a plane perpendicular to the parallel lines. [C. U. 1926.]

7. Find the locus of the foot of the perpendicular drawn from a given point upon any plane passing through a given st. line. [D. B. 1924.]

8. From an external point P, PO is drawn perpendicular to the plane XY and LM is any st. line in the plane XY. If PQ be drawn perpendicular to LM, show that OQ is also perpendicular to LM.

Proposition VI.

(*i*) *Of all straight lines drawn from an external point to a plane, the perpendicular is the shortest.**

(*ii*) *Of obliques drawn from a given point, those which cut the plane at equal distances from the foot of the perpendicular are equal.*

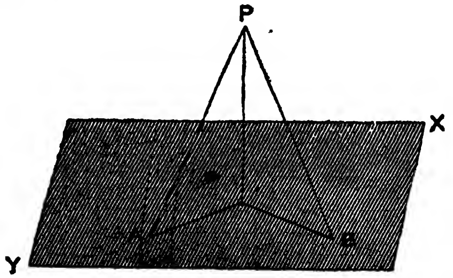


Fig. 7

*C. U. 1914, '15, '27, '31, '35. †[C. U. 1931.]

(i) Let PO be the perpendicular to the plane XY from the external point P and let PA be any oblique meeting the plane at A.

It is enough to prove that PO is less than PA.

Join OA. Since PO is perpendicular to the plane XY, it is also perpendicular to OA, so that the $\angle POA$ is a rt. angle.

Hence in the $\triangle POA$, the $\angle PAO$ is less than the $\angle POA$. $\therefore$ PO is less than PA.

That is, the perpendicular is less than any oblique.

Hence the perpendicular PO is the shortest.

(ii) Let the obliques PA and PB cut the plane XY at equal distances OA, OB from the foot O of the perpendicular PO.

It is required to prove that $PA = PB$.

Join OA, OB. Since PO is perpendicular to the plane XY, it is also perpendicular to OA, OB, so that each of the angles POA, POB is a right angle.

Now in the $\triangle$s POA, POB, we have

$OA = OB$, PO common and the $\angle POA =$ the $\angle POB$.

$\therefore$ the $\triangle$s are congruent, so that $PA = PB$.

EXERCISES.

1. Equal straight lines drawn to meet a plane from a point outside it are equally inclined to it.

2. Equal straight lines drawn to meet a plane from a point outside it are equally inclined to the perpendicular drawn from the point to the plane.

3. Find the locus of the feet of equal obliques drawn from an external point to a plane. [C. U.]

4. Draw a straight line equally inclined to three straight lines which meet in a point, but are not in the same plane.

5. Of obliques drawn from an external point to a plane, those which cut the plane nearer to the foot of the perpendicular are less than those which cut the plane at a greater distance and conversely.

6. If three points A, B, C on a plane are equidistant from an external point O, show that the foot of the perpendicular drawn from O to the plane is the centre of the circle through A, B, C. [C. U. 1914.]

7. If through the in-centre or an ex-centre of a triangle ABC a normal is drawn to the plane of the triangle, show that any point on the normal is equidistant from the sides of the triangle.

8. Find the locus of points equidistant from the angular points of a regular polygon.

9. From the centre of a circle circumscribed about a plane polygon a normal is drawn to the plane of the polygon. Show that any point on this normal is equidistant from the angular points of the polygon.

10. Perpendiculars are drawn from an external point to a system of coplanar straight lines passing through a fixed point. Find the locus of the feet of these perpendiculars. [Join the external pt. P to the fixed point O at which the st. lines intersect. The distance of the foot of any perpendicular from the mid-pt. of OP is constant. Hence the required locus is a circle.]

Definition. *The* **projection** *of a line on a plane is the locus of the foot of the perpendicular drawn from any point in the given line to the given plane.*

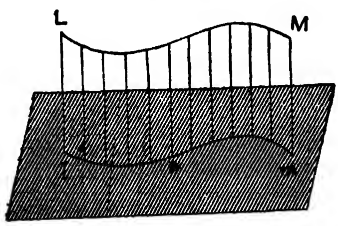


Fig. 8

In the adjoining figure from all points on LM perpendiculars are drawn on the plane XY. The line lm, being the locus of the feet of these perpendiculars is the projection or *orthogonal* projection of LM on the plane XY.

PROPOSITION VII.*

The projection of a straight line on a plane is itself a straight line.

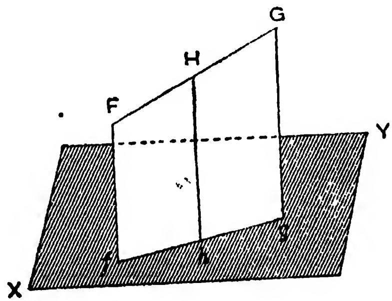


Fig. 9

Let FG be any straight line and H any point on it. From H draw Hh perpendicular to the given plane XY to meet it in h.

It is required to prove that the locus of h is a st. line.

Draw Ff, Gg perpendiculars to the plane XY meeting it in f and g respectively.

Now Ff, Hh, Gg being all perpendicular to the same plane XY, are parallel to one another.

Again, these parallels are cut by the st. line FG.

$\therefore$ they are coplanar.

*C. U. 1916, '18, '20, '23, '26, '30, '34.

$\therefore$ the point h is in the line of section of the planes XY and Fg, so that h is in the st. line fg.

But H is any point in FG.

$\therefore$ h is any point on the projection of FG.

Hence the projection of FG is the st. line fg in the plane XY.

Cor. A straight line and its projection on any plane are coplanar.

Note. The angle between a straight line and a plane is *the angle between the straight line and its projection* on the plane.

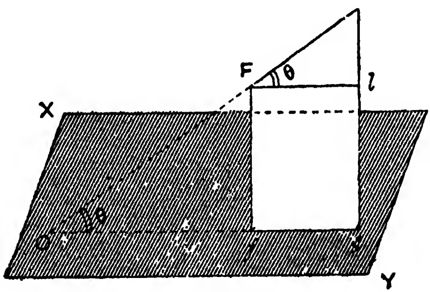


Fig. 10

In order to find the angle between the st. line FG and the plane XY, let us determine the angle between FG and its projection fg on XY in the adjoining figure.

Since FG and its projection fg are coplanar, they must meet at some point O in XY, when produced if necessary. Then the angle between FG and the plane XY is measured by the angle GOg.

EXERCISES.

1. Find the length of the projection of a straight line on a plane in terms of the length of the line and the angle which it makes with the plane. [C. U. 1934.]

Let fg be the projection of the st. line FG on the plane XY (as shown in fig. 10). Produce FG and gf, if necessary, to meet at O in XY. Through F draw Fl parallel to fg to meet Gg at l. Now the $\angle GFl=$ the $\angle GOg=\theta$, where θ is the angle between the st. line FG and the plane XY.

Hence the length of the projection of FG on XY

$$=fg=Fl=FG\cos\theta.$$

Note. The projection is maximum, when $\cos\theta=1$, *i.e.*, $\theta=0$,

i.e., when FG is parallel to the plane XY;

when FG is perp. to the plane XY, $\theta=90°$. $\therefore$ $\cos\theta=0$,

so that the projection, $FG\cos\theta=0$.

Thus the projection of a line on a plane can not be greater than the length of the line. The projection increases as θ decreases and **the maximum value of the projection is equal to the length of the line.**

2. Equal obliques drawn from a point to a plane have equal projections on that plane.

3. **If a str. line outside a given plane is parallel to any straight line in the plane, it is parallel to the plane itself.**

[C. U. 1917, '31, '33.]

[Let the st. line FG outside the plane XY be parallel to the st. line PQ drawn in the plane XY. *It is required to prove that FG is parallel to the plane* XY. Draw a plane passing through the parallel st. lines FG, PQ. If FG be not parallel to the plane XY, it will meet XY at some pt. on the line of section PQ. But by hypothesis, FG and PQ are parallel. $\therefore$ FG, can never meet the plane XY, *i.e.*, FG is parallel to XY.]

4. Prove that parallel straight lines have parallel projections on a plane. Is there any exception to this ?

5. Equal and parallel straight lines have equal and parallel projections on a plane. [C. U. 1923.]

6. A straight line in space has equal projections on parallel straight lines. [*See Ex. 1.*]

7. *The angle which a straight line makes with its projection on a plane is less than that which it makes with any other straight line which meets it in that plane.* [C. U. 1918, '30, '31.]

[Let ab be the projection of AB, in the plane XY. If BA meets its projection ba at O in XY, through O draw any other st. line OC in XY, so that $OC = Ob$. Now Bb being a perpendicular, $BC > Bb$.]

8. If a straight line is parallel to a plane, show that it is parallel to its projection on that plane.

9. *Show that if the projections of a given line on two intersecting planes be both straight lines, the given line is itself a straight line.* [*The given line becomes the lines of intersection of two planes.*]

[C. U. 1926.]

10. *The projection of the middle point of a straight line on a plane is the middle point of its projection.* [C. U. 1916.]

11. If FG be divided at any point H in a given ratio, show that h, the projection of H divides fg the projection of FG in the same ratio.

12. *Show that the line of intersection of two planes which respectively pass through two parallel straight lines is a third straight line parallel to either of the given straight lines.*

[*Apply Ex. 3.*] [C. U. 1922, '35.]

13. *A straight line AB is parallel to each of two intersecting planes ; show that AB is also parallel to their line of section.*

[*See Ex. 3.*] [C. U. 1917, '33.]

14. *Show that through a given point a plane may be constructed parallel to each of two skew lines.* [*Through the given pt. draw two lines parallel to the skew line.* (*Ex.* 3.) [C. U. 1931.]

15. Through either of two skew lines a plane may be made to pass to which the other line is parallel. [Through any point in one of the skew lines draw a line parallel to the other skew line. These two lines determine a plane. (*Ex.* 3)

16. *If, of the three lines of intersection of three planes, two be parallel, show that the third will also be parallel to the other two.*

[*Ex. 12.*] [C. U. 1914.]

Definitions :

1. *The angle between two skew lines is the angle between one of them and a line drawn through any point on it parallel to the other.*

2. When two planes intersect they are said to form a **dihedral angle.** [C. U. 1934.]

3. **The dihedral angle between two planes is measured** as the plane angle between the two straight lines drawn from any point in their line of intersection at right angles to it, one in each plane. [C. U. 1934.]

In figure 10(a), the planes LN and MP intersecting along LM from a dihedral angle between them. From any point O in LM, straight lines, OA, OB are drawn perpendicular to LM, in the planes LN and MP respectively. Then the angle AOB (in the plane AOB) measures the dihedral angle between the planes LN and MP.

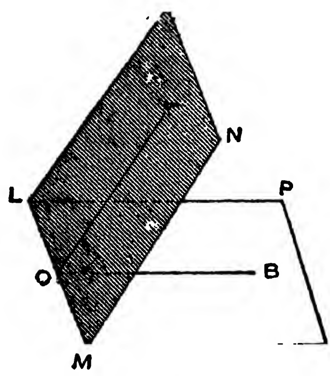


Fig. 10(a)

4. **One plane is said to be perpendicular to another,** when the dihedral angle between them is a right angle. [C. U. 1928.]

The plane LN will be perpendicular to the plane MP, if the dihedral angle AOB be a right angle.

PROPOSITION VIII.

*If a straight line is perpendicular to a plane, then any plane passing through the perpendicular is perpendicular to the given plane.**

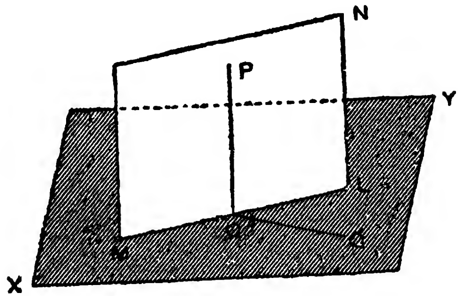


Fig. 11

Let the straight line OP be perpendicular to the plane XY and let MN be any plane passing through the perpendicular OP.

It is required to prove that the plane MN is perpendicular to the plane XY.

Let OQ be drawn in the plane XY perpendicular to ML, the line of intersection of the planes MN and XY. Since OP is perp. to the plane XY, it is perp. to OQ and the line of intersection MOL, so that the $\angle POQ$ is a right angle.

Again, since OP, OQ are both perp. to the line of intersection ML, the $\angle POQ$ measures the dihedral angle between the planes XY, MN.

Thus the dihedral angle POQ is a right angle.

Hence the plane MN is perp. to the plane XY.

*C. U. 1919, '37, '39, '41.

Cor. 1. If from any point on the line of intersection ML of two perpendicular planes XY and MN, a line be drawn in either of the planes perpendicular to the line of intersection ML, then it will also be perpendicular to the other plane.

Cor. 2. If from any point in either of the planes XY and MN which intersect at right angles along the st. line ML, a straight line be drawn perpendicular to the other, then it will lie wholly in the first plane.

Proposition IX.

*If two intersecting planes are each perpendicular to a third plane, their line of section also is perpendicular to that plane.**

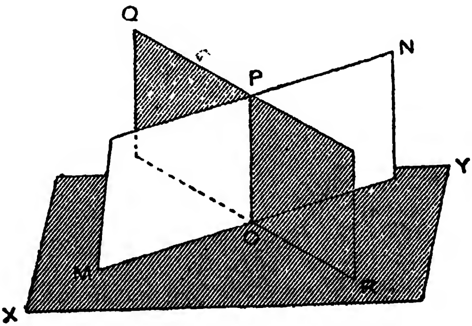


Fig. 12

Let the two planes MN and QR which intersect in the st. line OP be both perpendicular to the plane XY.

It is required to prove that PO is perpendicular to the plane XY.

From any point P, common to the two planes MN and QR, let a perpendicular be drawn to the plane XY. Since the planes MN and QR are both perpendicular to the plane

*C. U. 1910, '17, '20, '29, '43.

XY, the perpendicular through the common piont P must lie wholly in each of the planes MN and QR.

[*Prop. VIII, Cor. 2*]

Hence the perpendicular through P must be the st. line common to both the planes MN and QR, *i.e.*, it must coincide with their line of intersection PO.

Hence PO is perpendicular to the plane XY.

EXERCISES.

1. Show that the dihedral angle between two intersecting planes is equal or supplementary to the rectilinear angle between their normals. [C. U. 1910.]

2. If from any point two normals are drawn to two intersecting planes, show that the line of section is perpendicular to the plane of the normals.

3. *Shew that three mutually perpendicular planes cut one another in three mutually perpendicular straight lines.* [C. U. 1929.]

4. *Show that a plane can be drawn perpendicular to a given plane and passing through any given st. line.* [C. U. 1917, '19, '20.]

5. *If the intersections of three planes be parallel to one another, the normals drawn to these planes from any point are coplanar.*

[*See Ex. 2.*] [C. U. 1910.]

6. *If a plane intersects two parallel planes, show that the corresponding dihedral angles are equal.* [C. U. 1937.]

7. Show that a straight line is equally inclined to any number of parallel planes which it intersects.

8. Through a given point draw a plane perpendicular to each of two intersecting planes.

9. *Show that the planes bisecting the sides of a triangle at right angles meet in a straight line perpendicular to the plane of the triangle.*

10. If two planes be perpendicular to each other, show that every straight line perpendicular to one plane is either parallel to or lies on the other.

SOLID ANGLES.

1. When three or more planes mutually intersect in straight lines which meet in a point, they are said to form a **solid angle** at the point.

The point at which the planes meet is called the **vertex** of the solid angle. In fig. 13, the vertex is the point S. The solid angle is denoted by (S, ABC). [C. U. 1924.]

The lines of intersection of consecutive planes are called the **edges** of the solid angle. In fig. 13, the st. lines SA, SB, SC are the edges of the solid angle (S, ABC). The plane angle formed by consecutive edges at the vertex of a solid angle are called its **face-angles.** In fig. 13, the angles ASB, BSC, CSA are the face-angles.

The angles between the consecutive planes forming a solid angle are called its dihedral angles.

2. The solid angle formed by the mutual intersection of three planes is called a **trihedral angle.** The solid angle (S, ABC) shown in fig. 13 is a trihedral angle.

The solid angle formed by the mutual intersection of *more than three planes* at a point is called a **polyhedral angle.**

When the number of planes forming a solid angle is four, the solid angle is called a **tetrahedral angle.** When the number of planes is five, the solid angle is called a **pentahedral angle,** and so on.

3. When the section of the faces of a solid angle by any plane becomes a polygon having no re-entrant angles, the solid angle is said to be **convex.**

Proposition X.*

*In a trihedral angle the sum of any two of the face-angles is greater than the third.**

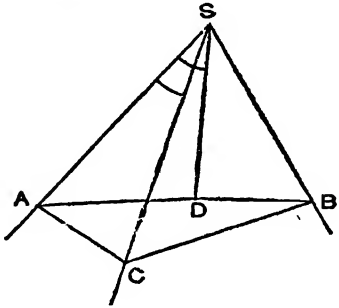


Fig. 13

Let (S, ABC) be a trihedral angle and let the $\angle ASB$ be the greatest of the three face-angles, namely, ASB, BSC, CSA.

Since the $\angle ASB$ is the greatest, the sum of the $\angle$s ASB, BSC is evidently greater than the $\angle ASC$, and the sum of the $\angle$s ASC, ASB is also evidently greater than the $\angle BSC$.

It is therefore enough to prove that the sum of the $\angle$s ASC, CSB is greater than the $\angle ASB$.

In the plane ASB make the $\angle ASD$ equal to the angle ASC and make $SD = SC$.

Draw a st. line through D in the plane ASB cutting SA, SB at A, B respectively. Join AC, CB.

Now in the Δs ASC, ASD, we have

$SC = SD$, AS common and the $\angle ASC =$ the $\angle ASD$.

$\therefore$ the Δs are congruent, so that $AD = AC$.

---

* C. U. 1909, '11, '18, '22, '24, '32.

Again, from the $\triangle ABC$,

$$AC + CB > AB, \text{ i.e., } > AD + DB.$$

$$\therefore \quad CB > DB.$$

Now, in the $\triangle$s BSC, BSD,

since $SD = SC$, SB is common to both,

and $CB > DB$,

$\therefore$ the $\angle BSC$ is greater than the $\angle BSD$.

Hence the sum of the $\angle$s ASC, CSB is greater than the sum of the $\angle$s ASD, DSB, *i.e.*, greater than the $\angle ASB$.

EXERCISES.

1. *Prove that the sum of the interior angles of a skew quadrilateral is less than four right angles.* [C. U. 1913, '32.]

2. In a trihedral angle the difference of any two face-angles is less than the third. [The $\angle BSC >$ the $\angle BSD$ in fig. 13.]

3. *OA, OB, OC are three intersecting straight lines such that the angle BOC is equal to the sum of the angles AOB, AOC; show that OA, OB, OC are coplanar.* [C. U. 1912, '22, '35.]

[If not, OA, OB, OC will form a trihedral angle and hence etc.]

4. Prove that three planes in general meet in a point. Discuss the three exceptional cases.

5. *The sum of any three face-angles of a tetrahedral angle is greater than the fourth.* [C. U. 1911.]

Proposition XI.

In a convex solid angle the sum of the face-angles is less than four right angles.

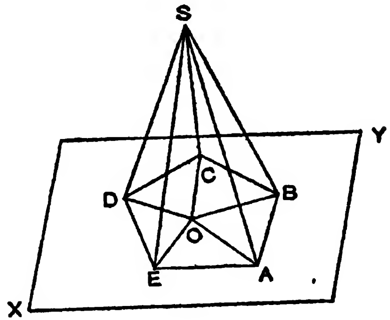


Fig. 14

Let (S, $ABCDE$) be a convex solid angle, of which ASB, BSC, CSD, DSE, ESA are the face-angles.

It is required to prove that the sum of the face-angles ASB, BSC, CSD, DSE, ESA *is less than four right angles.*

Let a plane XY cut the planes of the face-angles in st. lines which form the convex polygon $ABCDE$. Take a point O within the polygon $ABCDE$ and join OA, OB, OC, OD, OE.

In the trihedral angle A,

the $\angle SAB$ + the $\angle SAE$ is greater than the $\angle BAE$, [*Prop. X.*]

i.e., greater than the $\angle OAE$ + the $\angle OAB$.

Similarly for each of the trihedral angles at B, C, D, E.

$\therefore$ the sum of the base angles of the Δs with vertex S

* C. U. 1910, '27.

is greater than the sum of the base angles of the Δs with vertex O.

Since the number of triangles with vertex S is equal to the number of triangles with vertex O, the sum of all the angles of the triangles with vertex S is equal to the sum of all the angles of the triangles with vertex O. Hence the sum of the face-angles at S is less than the sum of the angles at O.

But the sum of the angles at O is four right angles ;

$\therefore$ the sum of the face-angles at S is less than four right angles.

EXERCISES.

1. *If three straight lines drawn from a point in space make with each other three angles whose sum is four right angles, the three lines are then caplanar.* [C. U. 1910.]

[If not, the three st. lines will form a solid angle, the sum of whose face-angles is less than four rt. $\angle$s.]

2. The sum of the three face-angles of a trihedral angle is less than four right angles.

3. If the angular points of a convex polygon be joined to any point so that the sum of the angles between consecutive lines is equal to four right angles, show that the straight lines are coplanar.

MISCELLANEOUS EXAMPLES I.

1. Describe a method of drawing the line of greatest slope through any point on an inclined plane. (The line of greatest slope is the st. line making the greatest angle with the horizonal plane.)

2. A plane cuts two sides of a triangle proportionally. Show that it is parallel to the third side.

3. *Determine the locus of a point equidistant from three non-coplanar intersecting st. lines.*

4. If *LM, PQ* be two skew st. lines, prove that the lines joining their extremities *viz. LP, MQ* and *LQ, MP* are also skew.

5. On what planes will the projections of two given skew lines be perpendicular ?

[Draw two perpendicular planes, each passing through one skew line. Any plane perpendicular to the line of section satisfies the condition.]

6. On what planes will the projections of two given skew lines be parallel ? [*See Prop. VII, Ex. 15.*]

[Draw two parallel planes each passing through one skew line. Any plane perpendicular to the parallel planes satisfies the condition.]

7. If *SX* be a straight line within the solid angle (*S, ABC*) determined by *SA, SB, SC*, show that

(i) the sum of the angles *ASX, BSX, CSX* is greater than half the sum of the angles *ASB, BSC, CSA*.

(ii) The sum of the angles *ASX, CSX* is less than the sum of the angles *ASB, CSB*.

(iii) the sum of the angles *ASX, BSX, CSX* is less than the sum of the angles *ASB, BSC, CSA*.

8. ***Show that in a trihedral angle the sum of the dihedral angles i.e. the angles between the faces is greater than*** **180°.** [***From any point within the solid angle draw perpendiculars to the three faces giving rise to a trihedral angle.*** [*See Ex. 1, 2 ; Prop. IX.*]

9. There can not be more than five regular polyhedra.* [C. U.]

[The least number of plane surfaces required to form a solid angle is three. Consequently the minimum number of plane angles required to form a solid angle is three, and the sum of the three plane angles must be less than 360°, so that each face-angle must be less than 120°. Since the polyhedron is regular, all the face-angles are equal. Again, each angle of a regular hexagon is ≯120° and the angles of regular figures with more than six sides are still greater. Hence regular figures of more than five sides are excluded. In other words, only

***A polyhedron is a solid figure bounded by plane surface only.**

equilateral triangles, squares and regular pentagons can form the faces of a regular polyhedron.

Thus when the faces of a regular polyhedron are equilateral triangles, the number of such triangles forming a solid angle may be (i) *three*, because $3 \times 60^\circ = 180^\circ$; (ii) *four*, because $4 \times 60^\circ = 240^\circ$; (iii) *five*, because $5 \times 60^\circ = 300^\circ$ which is evidently less than 360°. But more than five such triangles can not form a solid angle of a regular polyhedron by Prop. XI.

When the faces are squares, the number of such squares forming a solid angle of a regular polyhedron be (iv) *three* only, because $3 \times 90^\circ = 270^\circ$ which is less than 360°, but $4 \times 90^\circ \nless 360^\circ$.

When the faces are regular pentagons, the number of such regular pentagons forming a solid angle of a regular polyhedron may be (v) only *three* but not more than three, because $108^\circ \times 3 = 324^\circ$. Hence the number of regular polyhedra is only five.]

10. *Show how to draw a st. line through a given point in space intersecting two given skew lines.* [C. U. 1912.]

[Draw a plane passing through the given pt. and one of the skew lines. Produce the plane to meet the other skew line at some point.]

11. Prove that all points in space equidistant from two given points lie in a plane. [*See Prop. III, Ex. 1.*] [C. U. 1939.]

12. *Draw a straight line perpendicular to two given skew straight lines. Show that the shortest distance between two non-intersecting straight lines is a straight line perpendicular to both.* [C. U. 1912.]

[Let AB, CD be two skew lines. Through any point P in AB, draw PQ parallel to CD. At P draw PR perp. to PB and PQ. Produce the plane passing through AB, PR to meet CD at S. Through S draw ST parallel to PR meeting AB in T.]

13. *Show that, through a given point P, a plane may be constructed parallel to each of two lines AB and CD, which do not lie in the same plane.* [C. U. 1931.]

[Through the given point P draw two lines parallel to AB and CD.]

14. OA, OB, OC are three concurrent lines, each of which is perpendicular to the other two ; show that

(i) if OX, OY, OZ are perpendicular to BC, CA, AB respectively, XYZ is a pedal triangle of the triangle ABC ;

(ii) if OP is perpendicular to the plane of ABC, show that P is the orthocentre of the triangle ABC.

15. Show that any plane passing the middle points of three non-coplanar lines PQ, QR, RS is parallel to PR and QS.

[*Prop. VII. Ex. 3.*] [D. B. 1927.]

16. If there be two parallel planes, show that any straight line drawn in any one of the planes is parallel to the other.

17. *Show that if AB and CD be two parallel straight lines and O any point outside the plane containing both, then the planes OAB and OCD intersect in a straight line parallel to both.*

[*See Prop. VII, Ex. 12.*] [C. U. 1929, '35.]

18. Show that from a given external point, not more than one perpendicular can be drawn to a plane. [C. U. 1927.]

19. Show that if three planes mutually intersect, their three lines of intersection will be either concurrent or parallel. [C. U. 1911, '24.]

[Let X, Y, Z be three non-collinear planes. If the line of intersection of any two planes X, Y be not parallel to the third plane Z, it will intersect it at some point P. $\therefore$ P being common to the planes X and Z must lie on their line of section. For the same reason P must lie on the line of intersection of the planes Y and Z. Thus three planes will be concurrent, etc.]

20. From O, the centre of a circle, a perpendicular OA is erected to the plane of the circle. Show that all points on the circumference are equidistant from A. [C. U. 1937.]

CHAPTER II.

SOLID FIGURES.

1. Any portion of space bounded by one or more surfaces, plane or curved is called a **solid figure** or a **solid.**

The bounding surfaces, plane or curved, are called the **faces** of the solid.

The lines in which the adjacent faces intersect are called its **edges.**

2. A polyhedron is a solid figure bounded by plane faces only.

Note. In order that a polyhedron may be **regular**, all its faces must be regular figures and the same number of faces must meet at each vertex.

THE REGULAR POLYHEDRA.*

(i) A regular polyhedron of which each solid angle is formed by three equilateral triangles is called a regular **tetrahedron.**

In the adjoining figure the equilateral triangles ABC, ACD, ABD, BCD are its four faces. The points A, B, C, D are its vertices and the straight lines AB, AD, AC, BC, CD, DB are its six edges.

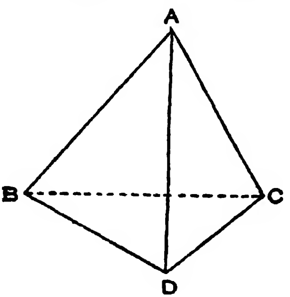


Fig. 15.

* See Miscellaneous Examples I, Ex. 9.

(ii) A regular polyhedron of which each solid angle is formed by four equilateral triangles is called a regular **octahedron.**

It has six vertices and at each vertex a solid angle is formed by four equilateral triangles, so that the number of faces is eight and edges is twelve.

(iii) A regular polyhedron of which each solid angle is formed by five equilateral triangles is called a regular **icosahedron.**

It has twelve vertices and at each vertex a solid angle is formed by five equilateral triangles, so that the number of its faces is twenty and edges thirty.

(iv) A regular polyhedron of which each solid angle is formed by three squares is called a **cube.**

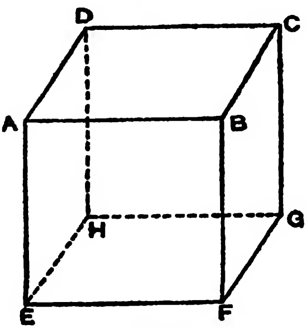


Fig. 16

In the adjoining figure the squares $ABCD$, $EFGH$, $ADHE$, $DHGC$, $BCGF$, $ABFE$ are its six faces. The points A, B, C, D, E, F, G, H are its eight vertices and AB, BC, CD, DA, EF, FG, GH, HE, AE, DH, BF, CG are its twelve edges.

(v) A regular polyhedron of which each salid angle is formed by three regular pentagons is called a regular **dodecahedron.**

It has twenty vertices and at each vertex a solid angle is formed by three regular pentagons, so that the number of faces is twelve and edges is thirty.

3. A parallelopiped is a solid figure bounded by three pairs of parallel plane surfaces.

Each of the six plane surfaces bounding a parallelopiped is a parallelogram, so that the opposite faces are equal in all respects and the twelve edges fall into three groups, each group consisting of four edges which are equal and parallel to one another.

4. A parallelopiped of which the faces are rectangular is called a rectangular parallelopiped, or rectangular solid or cuboid.

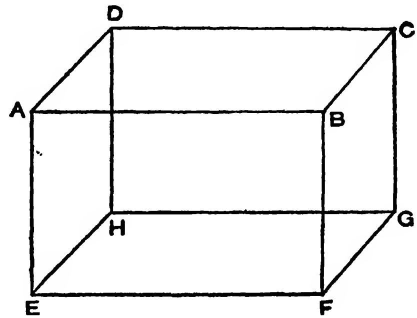


Fig. 17

The six faces $ABCD$, $EFGH$, $ADHE$, $DHGC$, $BCGF$, $ABFE$ are rectangles inclined at right angles to one another.

5. A rectangular parallelopiped whose all the six faces are squares is called a **cube**. [*See fig. 16.*]

6. To find the surface and volume of a rectangular parallelopiped.

From fig. 17, we have $AB = DC = HG = EF = a$,

$$AD = BC = EH = FG = b,$$

and $$AE = DH = BF = CG = c,$$

taking a, b, c to be the length, breadth and height of the rectangular parallelopiped.

Area of the rectangle $ABCD = AB \times AD = ab$ units of area.

Area of the rectangle $EFGH = EF \times EH = ab$ " " .

Area of the rectangle $ADHE = AD \times DH = bc$ units of area.

Area of the rectangle $BCGF = BC \times GC = bc$,, ,, .

Area of the rectangle $DCGH = DH \times HG = ac$,, ,, .

Area of the rectangle $ABFE = AB \times AE = ac$,, ,, .

$\therefore$ area of the total surface of the rectangular parallelopiped $= 2ab + 2bc + 2ca$ units of area ... (i).

The volume of the rectangular parallelopiped

$= EF \times EH \times AE = abc$ units of volume ... (ii).

To find the total surface of a cube, put $a = b = c$ in (i), because in a cube all the twelve edges are equal.

Hence the area of each face of a cube $= a^2$ units of area.

$\therefore$ the area of the total surface of the cube $= 6a^2$ units of area, and the volume of the cube $= a^3$ units of volume.

7. Right Prism.

A **prism** is a solid bounded by plane faces of which the side-faces are parallelograms and the two ends are congruent polygons in parallel planes.

The lines of intersection of consecutive side-faces are called the side-edges of a prism. For every prism the side-edges are parallel and equal.

The two ends of a prism may be triangles, quadrilaterals or polygons of any number of sides. A prism is said to be triangular, quadrilateral, or polygonal, according as the end-faces (or ends) are triangles, quadrilaterals or polygons.

When the side-edges of a prism are perpendicular to its ends the prism is said to be **right.**

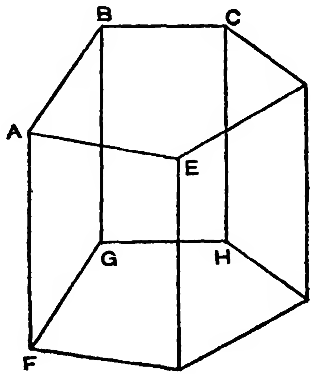
In the case of a *right prism* the side-faces are rectangles.

Thus a **right prism** is defined to be a solid bounded by plane faces of which *the side-faces are rectangles* and the two ends are congruent polygons in parallel planes.

[C. U. 1937.]

All other prisms whose side-edges are not perpendicular to the ends are called **oblique** prisms.

In fig. 18, $ABGF$, $BCHG$, $CDKH$, $EDKL$, $AELF$ are rectangular side-faces perpendicular to the parallel end-faces $ABCDE$, $FGHKL$.



Right Prism. Fig. 18

8. To find the area of the side-faces and the volume of a right prism.

If the height or the perpendicular distance between parallel end-faces be h units of length and the sides AB, BC, CD, DE and EA be a, b, c, d and e units of length respectively, then the areas of the rectangular side-faces are given below :

$$AG = ah,\ BH = bh,\ CK = ch,\ DL = dh \text{ and } EF = eh.$$

Hence the **total area of the side-faces of any right prism**

$$= ah + bh + ch + dh + eh = (a + b + c + d + e)h$$

$$= (\text{perimeter of the base}) \times \text{length of the side-edge},$$

because in the case of a right prism, length of the side-edge is equal to the height of the prism.

Similarly in the case of an oblique prism, the **total area of the side-faces of the oblique prism**

= (perimeter of the base) × height,

where height is the perpendicular distance between the parallel ends, because each side-face is a parallelogram.

The **volume of any right prism**

= (area of the base) × side-edge

i.e. (area of the base) × height,

because in the case of a right prism height is equal to the length of the side-edge.

The **volume of an oblique prism**

= (area of the base) × (height),*

where height is the perpendicular distance between the parallel ends.

N. B. *Area of the whole surface = the area of the side-faces + the area of the end-faces.*

9. Square and triangular pyramids.

A pyramid is a solid bounded by plane faces, of which the face representing the base is any polygon and the remaining faces are triangles meeting at a point called the **vertex.**

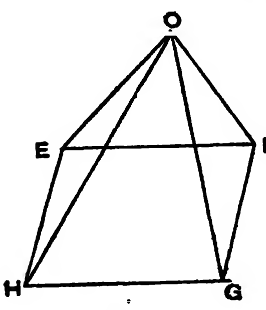


Fig. 19

In the adjoining fig., O is the vertex at which the triangular faces OEH, OHG, OGF, OEF meet, the figure $EFGH$ is the base, and the st. lines OE, OH, OG, OF, EF, FG, GH, HE are the **edges.**

*A parallelopiped is a special form of oblique prism. Cubes and rectangular parallelopipeds are special forms of right prisms.

A pyramid is said to be *triangular*, *quadrilateral* or *polygonal* according as its base is a triangle, a quadrilateral or a polygon.

If the base of a pyramid be a regular polygon and if its vertex lie on a straight line perpendicular to its base and passing through the centre of the base (*i.e.* the centre of its inscribed or circumscribed circle), the pyramid is said to be **right.**

Thus in the case of a right pyramid, the side-edges are all equal and the lateral faces are all congruent isosceles triangles.

All other pyramids are **oblique.**

A **triangular pyramid** is a pyramid standing on a triangular base. It is bounded by four triangular faces. In fig. 15, $ABCD$ is a triangular pyramid. A **tetrahedron*** is a pyramid on triangular base, *i.e.* it is triangular pyramid. [*See fig. 15*]

A **square pyramid**† is a pyramid standing on a square base.

In fig. 19, $OEFGH$ is a square pyramid of which the base $EFGH$ is a square and the plane faces OEH, OHG, OGF, OFE are triangles meeting at the vertex O.

*A *tetrahedron or a triangular* pyramid is said to be **right,** if its base is an equilateral triangle and its other three lateral faces are congruent isosceles triangles.

† A *square pyramid* is said to be **right,** if its four lateral faces are congruent isosceles triangles, while the base is a square.

10. To determine the area of the slant surface and the volume of a pyramid.

(i) The total area of the slant surface of a pyramid on a polygon as base = the sum of the areas of the triangular side-faces meeting at the vertex.

The area of the whole surface of a pyramid = the total area of the slant surface + the area of the base.

(ii) In the case of a *right pyramid* standing on a regular polygon of n sides as base, the total area of the slant surface = the sum of the areas of n congruent isosceles triangles = $\frac{1}{2}n$ base × altitude of any isosceles triangle, $= \frac{1}{2}nab$, where b = base of the triangle, and a = altitude of the triangle or slant height of the right pyramid.

Hence **the total area of the slant surface of a right pyramid** = $\frac{1}{2}$ (perimeter of the base) × slant height, where slant height means the perpendicular distance of the vertex from the base of the triangle *i.e.* altitude of the triangle.

The volume of a pyramid on any base

$$= \tfrac{1}{3}(\text{area of the base}) \times \text{height of the pyramid},$$

where the height of the pyramid means the perpendicular distance of the vertex of the pyramid from its base.

11. The right circular cone.

A **right circular cone** is a solid generated by the complete revolution of a right-angled triangle about one of its sides containing the right angle, as axis. [C. U. 1937.]

In the adjoining figure, POQ is the right-angled triangle which by its complete revolution about the side PO generates the cone. The circle with centre O and radius OQ $(=r)$ descibed by the other side OQ is the **base** of the cone. The point P is called the **vertex,** the line OP is called the **axis** and the hypotenuse PQ is called the **generating line** of the cone.

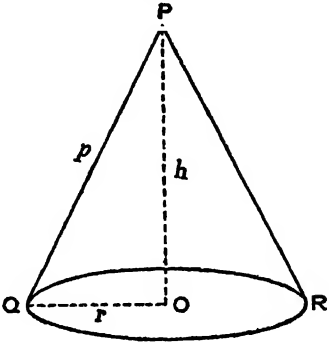


Fig. 20.

The angle QPR is called the vertical angle, the angle QPO is called the semi-vertical angle, and the length of the axis OP $(=h)$ is termed the height of the cone.

Note. *A right circular cone may be regarded as a right pyramid standing on a circular base.*

In general a cone is *generated* by the motion of a straight line, such that it always passes through a fixed point and slides continuously over a fixed curve (not coplanar with it).

The fixed curve is called the guiding curve. In fig. 20, the circle QR is the guiding curve, P is the fixed point (called the vertex), and PQ is the moving line (called the generating line).

12. To find the surface and the volume of a right circular cone.

In fig. 20, the slant height of the cone $= PQ = p$; radius of the base $= r$, and the height of the cone $= OP = h$.

(i) The total **curved surface** of the cone

$= \frac{1}{2}$ (circumference of the base) $\times$ (slant height)

$= \frac{1}{2}(2\pi r)p = \pi rp$ units of area. [*Art. 11, Note.*]

The **whole surface** = curved surface + area of the base

$= \pi rp + \pi r^2 = \pi r(p + r).$

(ii) The **volume** of the cone $= \frac{1}{3}$ (area of the base) $\times$ height

$= \frac{1}{3}\pi r^2 \times h = \frac{1}{3}\pi r^2 h$ units of volume.

13. The right circular cylinder.

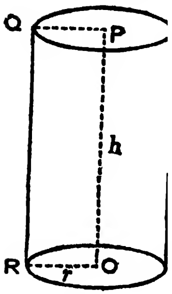


Fig. 21

A **right circular cylinder** is a solid generated by the complete revolution of a rectangle about one of its sides as axis.

In fig. 21, *OPQR* is a rectangle which revolves about its sides *OP* as axis, so that *QR* generates the cylinder.

The st. line *OP* is called the **axis** of the cylinder, its length h $(= OP)$ represents the **height** of the cylinder, and the st. line *QR* is called the **generating line** of the cylinder. The two plane circular **ends** described by the sides *OR*, *PQ* are called the **bases** of the cylinder.

In general, a **cylinder*** is generated by the motion of a straight line parallel to itself, so that it slides continually over a fixed curve (not coplanar with it).

14. To find the surface and the volume of *a right circular cylinder.*

*In a right circular cylinder the guiding curve is a circle, so that the base is circular and the generating line is perpendicular to the base.

(i) The area of the **curved surface** of a right circular cylinder

$$=(\text{circumference of the base}) \times \text{height}$$
$$= 2\pi r \times h = 2\pi rh \text{ units of area.}$$

The **whole surface** = curved surface + area of the circular ends

$$= 2\pi rh + 2\pi r^2 = 2\pi r(h+r).$$

(ii) The **volume** of the right circular cylinder

$$=(\text{area of the base}) \times \text{height}$$
$$= \pi r^2 \times h = \pi r^2 h \text{ units of volume.}$$

15. The Sphere.

A sphere is a solid generated by the complete revolution of a semi-circle about its diameter as axis.

In the adjoining figure the semi-circle PRQ revolves about the diameter PQ, so as to generate the sphere.

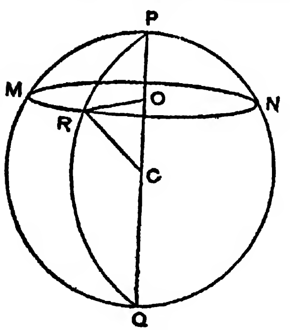


Fig. 22.

The surface of the sphere is the surface generated by the semi-circumference as it revolves completely about its diameter.

As the semi-circumference revolves about its diameter PQ, every point on it is at the same distance from its centre.

Thus the **surface** of a sphere is the locus of points in space whose distance from a fixed point is constant. The fixed point C is callad the **centre** and the constant distance CR the **radius** of the sphere.

A **diameter** of a sphere is any straight line passing through the centre and terminated bothways by the surface.

The total **area of the surface** of a sphere $=4\pi r^2$, where r is the radius of the sphere.

The **volume of a sphere** $=\frac{4}{3}\pi r^3$, where r is the radius of the sphere.

N. B. Thus the surface of a sphere is equal to four times the area of a circle of equal radius.

16. Any **plane section** of a sphere is a circle.

Let the plane RMN cut the sphere with centre C, and radius $CR(=r)$. [*See fig. 22.*]

Draw CO perp. to the cutting plane. Join CR, OR.

Now R being any point on the sphere and CO being perpendicular to OR, in the right-angled triangle COR, we have

$$OR^2=CR^2-CO^2=r^2-p^2, \text{ where } CO=p.$$

$$\therefore \quad OR=\sqrt{r^2-p^2}=\text{a constant.}$$

That is, the distance of any point on the line of section from O is a constant.

$\therefore$ the plane section of a sphere in a circle.

MISCELLANEOUS EXAMPLES II.

1. Find the surface and the volume of a sphere of radius 3 ft.

The area of the surface of the sphere

$=4\pi r^2=4\pi\times 3^2$ square feet

$=4\times 3{\cdot}1416\times 9$ sq. ft.* $=113{\cdot}0976$ sq. ft.

The volume of the sphere

$=\frac{4}{3}\pi r^3=\frac{4}{3}\pi\times 3^3$ cubic feet

$=36\times 3{\cdot}1416$ cu. ft. $=113{\cdot}0976$ cu. ft.

*$\pi=3{\cdot}14159\ldots\ldots=\frac{22}{7}$ roughly.

2. Find the area of the surface of a right prism, whose ends are squares of sides 3 inches, the height of the prism being one foot.

[C. U. 1936.]

The surface of a right prism = perimeter of the base $\times$ height.

But the perimeter of the base $= 3\times 4$ inches $= 1$ foot.

$\therefore$ the reqd. area of the surface $= 1\times 1 = 1$ sq. ft.

3. Find the volume and the area of the slant surface of a right circular cone of height 4 feet and the radius of whose base is 3 feet (π being equal to $\frac{22}{7}$). [C. U. 1939.]

The volume of the right circular cone $= \frac{1}{3}\pi r^2.h$ ... [*Art. 12.*]

$= \frac{1}{3}\times 3\cdot1416\times 3^2\times 4$ cu. ft.

$= 37\cdot6992$ cu. ft.

The slant surface of the cone $= \pi rp$ ... ... [*Art. 12.*]

$= 3\cdot1416\times 3\times 5$ sq. ft.

$= 47\cdot124$ sq. ft.,

$\therefore$ the slant height, $p = \sqrt{3^2+4^2} = 5$ ft.

4. A right circular cone 20 feet high has its upper part cut off by a plane passing through the middle point of its axis. If the plane of section be at right angles to the axis and if the radius of the original cone be 4 feet, find the volume of the truncated cone. [C. U. 1936]

The volume of the original cone $= \frac{1}{3}\pi r^2 h$ ... ... [*Art. 12.*]

$= \frac{1}{3}\pi\times 4^2\times 20$ cu. ft.

$= \frac{320}{3}\pi$ cu. ft.

The upper section of the cone by a plane perp. to its axis and passing through its mid.-pt. gives a right circular cone of radius 2 ft., height 10 ft.

$\therefore$ the volume of this cone $= \frac{1}{3}\pi\times 2^2\times 10$ cu. ft. $= \frac{40\pi}{3}$ cu. ft.

Hence the required volume of the truncated cone

$$= \frac{320\pi}{3} - \frac{40\pi}{3} = \frac{280\pi}{3} = \frac{280}{3}\times\frac{22}{7} = 293\frac{1}{3} \text{ cu. ft.}$$

5. Three golden solid spheres of radii 3, 4, 5 inches are melted into a single solid sphere. Find the radius of the single sphere.

6. Three solid glass balls of radii r, 6, 8 inches are melted into a solid sphere of radius 9 inches. Determine the value of r.

7. Two metallic solid spheres of radii R and r are melted into a single solid sphere. Determine the radius of the single sphere.

8. A solid spherical mass of copper is melted into a cylindrical solid rod of radius 6 cm. If the radius of the sphere be 3 cms., find the length of the rod.

9. The curved surface of a cylinder is 1000 sq. cms. and the diameter of the base is 20 cms.; find the volume of the cylinder. Also find its height to the nearest millimetre. [C. U. 1934.]

The circumference of the base of the cylinder $= 2\pi r = 20\pi$ cms.

$\therefore$ the height of the cylinder $= (1000 \div 20\pi) = \dfrac{50}{\pi}$ cms.

$$= \frac{50}{3 \cdot 1416} \text{ cm.} = 15 \text{ cm. } 9 \text{ mms. nearly.}$$

The volume of the cylinder $= \pi r^2 \times \dfrac{50}{\pi}$ cu. cms.

$$= 100\pi \times \frac{50}{\pi} = 5000 \text{ cu. cms.}$$

10. Find the area of the lateral surface and volume of a right prism of height 8 cms. and whose base is a regular hexagon of side 6 cms.

11. The height of a right prism is 10 inches and the sides of the triangular base are 9 inches, 10 inches and 17 inches. Determine the volume and the *whole* surface of the prism.

12. A rectangular block of copper of length 1 ft. 9 inches, breadth 1 ft., and height 11 inches is drawn out into a uniform wire of radius ·07 inch. Find the length of the wire.

13. A rectangular tank stands on a base of length 25 ft. and breadth 10 ft. If its height be 4 ft., find the quantity of water contained in it and determine the area of its inner surface.

14. If the length, breadth and height of a rectangular parallelopiped be a, b and c respectively, find the length of a diagonal. Hence show that the diagonals of a cuboid are equal, the length of a diagonal of a cube is equal to $\sqrt{3}$ times its side.

[In fig. 17, BF is perp. to the face EG. $\therefore$ BF is perp. to HF.

$\therefore$ $HB^2 = HF^2 + BF^2$.

Again, the angle HEF is a right angle,

so that $HF^2 = HE^2 + EF^2$.

Hence, $HB^2 = HE^2 + EF^2 + BF^2$.

That is any (diagonal)^2 = sum of the squares on the concurrent edges.

15. The diagonal of a cube is 20 cms. Calculate the length of the edge of the cube to the nearest millimetre.

16. The area of the whole surface of a cube is 486 sq. cms. Calculate its volume.

17. Find the ratio in which a plane drawn parallel to the base of a right circular cone divides its axis, so that the (i) volumes of the two parts are equal ; (ii) surfaces of the two parts are equal.

18. The diagonal of a rectangular solid is $5\sqrt{2}$ inches and the whole surface is 94 sq. inches. Find the sum of the length, breadth and height of the solid. [By ex. 14, $a^2+b^2+c^2=50$, and

$2(ab+ac+bc)=93$, where a, b, c are the edges of the solid.]

19. Find the volume and the whole surface of a regular* tetrahedron of edge s.

20. Find the (i) slant surface, (ii) the volume, of a right pyramid 12 cms. high, standing on a square base whose side is 10 cms.

21. Find the (i) slant surface, (ii) the volume of a right pyramid of height 6 cms., standing on an equilateral triangle of side 12 cms.

22. Find the volume of a pyramid 12 ft. high, standing on a rectangular base of length 10 ft. and breadth 8 ft.

23. Find the volume of a square pyramid 16 ft. high, standing on a square base whose side is 12 ft.

24. Determine the volume of a pyramid whose height is $10\sqrt{7}$ ft. and which stands on a triangle of sides 16 ft., 11 ft. and 9 ft.

[C. U. 1941.]

25. Find the length of a square pyramid of volume 576 cu. ft. which stands on a base 12 ft. square.

**A tetrahedron is said to be regular, if all the four faces are equilateral triangles.*

26. The upper portion of a right pyramid 30 ft. high, standing on a square base of side 16 ft. is cut off by a plane which is parallel to the base and passes through the middle point of its axis. Determine the volume of the truncated pyramid.

27. Show that the perpendicular from the vertex of a regular tetrahedron upon the opposite face is three times that drawn from its foot upon any other face.

28. The volume of a rectangular parallelopiped is 144 cu. inches and the areas of two side-faces are 36 sq. inches, and 12 sq. inches. Find the dimensions of the solid.

29. The volume of a rectangular parallelopiped is 1875 cu. cms. Find the length, breadth and height of the solid, if they be proportional to 5, 3, 1.

30. The diagonal of a rectangular paralellopiped is $6\sqrt{5}$ cms., its volume is 320 cu. cms., and area of its whole surface is 304 sq. cms. Calculate the dimensions of the solid.

31. The whole surface of a rectangular block is 214 sq. cms. The areas of two faces are 42 sq. cms., and 30 sq. cms., respectively. Find the edges of the block.

32. The edges of a cuboid are in the ratio 3 : 4 : 5. If the whole surface area be 1504 sq. cms., find the edges.

33. Show that the sum of the squares on the four diagonals of a rectangular parallelopiped is equal to the sum of the squares on the twelve edges.

34. Find the curved surface to the nearest sq. cms., and the volume to the nearest cu. cms., of a cylinder whose height is 16 cms. and radius of the base is 5 cms.

35. From a solid cylinder whose height 2·4 inches and radius ·7 inch, a conical cavity of the same height and base is cut out. Find the whole surface of the remaining solid to the nearest square inch.

36. The capacity of a cylindrical tumbler is fifteen times as great as that of (right circular) conical tumbler of equal base area. Compare their heights.

37. If V be the volume, S the area of the curved surface, h the height, r the radius of the base and α the semi-vertical angle of a right circular cone, show that $S=\frac{\pi h^2 \tan\alpha}{\cos\alpha}=\frac{\pi r^2}{\sin\alpha}$,

$$\text{and } V=\tfrac{1}{3}\pi h^3 \tan^2\alpha=\frac{1}{3}\cdot\frac{\pi r^3}{\tan\alpha}.$$

38. Show that the volume of a pyramid standing on a regular hexagon is $\frac{\sqrt{3}}{2}\cdot a^2 h$, where h is the height of the pyramid and a the length of a side of the regular hexagon.

39. Find the volume of a right prism of height h, standing on an equilateral triangle of side s.

40. A right pyramid stands on a rectangular base measuring 8 cms. by 6 cms. Find the height and volume of the pyramid, if each slant edge be of length 13 cms.

41. Find the lateral surface and the volume of a right circular cone 15 ft. high, the radius of whose base is 8 ft. [C. U. 1942.]

42. The faces of a tetrahedron are four equal equilateral triangles ; find the area of the faces of the tetrahedron, if the length of a side of each triangle is 4 ft. Find also the volume of the tetrahedron.

[C. U. 1938]

43. A right pyramid stands on a square base of side 12 ft. Find the height of the pyramid, if its volume is 576 cu. ft.

[*See Ex.* 25.] [C. U. 1943.]

44. A right prism stands on a triangular base whose sides are 17 cms., 10 cms., 9 cms., and the height is 10 cms. Find the volume and the whole surface. [C. U. 1940.]

45. Determine the surface of a sphere of radius 10 inches.

[C. U. 1936.]

ANSWERS

Miscellaneous Examples II.

5. 6 inches. 6. 1 inch. 7. $(R^3+r^3)^{\frac{1}{3}}$.

8. 1 cm. 10. 288 cms., $432\sqrt{3}$ cu. cms.

11. 360 cu. inches, 432 sq. inches. 12. 5000 yds.

13. 1000 cu. ft., 530 sq. ft. 14. $\sqrt{a^2+b^2+c^2}$.

15. 11·5 cms. 16. 729 cu. cms.

17. (i) $\sqrt[3]{2}-1 : 1$, (ii) $\sqrt{2}-1 : 1$. 18. 12 inches.

19. $\frac{s^3}{6\sqrt{2}}$, $\sqrt{3}s^2$. 20. 260 sq. cms., 400 cu. cms.

21. $72\sqrt{3}$ sq. cms., $72\sqrt{3}$ cu. cms. 22. 320 cu. ft.

23. 763 cu. ft. 24. 420 cu. ft.

25. 12 ft. 26. 2240 cu. ft.

28. 12 inches, 4 inches, 3 inches. 29. 25 cms., 15 cms., 5 cms.

30. 10 cms., 8 cms., 4 cms. 31. 7 cms., 6 cms., 6 cms.

32. 20 cms., 16 cms., 12 cms. 34. 503 sq. cms., 1257 cu. cms.

35. 18 sq. inches. 36. 5 : 1.

39. $\frac{\sqrt{3}}{4}s^2h$. 40. 12 cms. ; 192 cu. cms.

41. 427·26 sq. ft. nearly ; 1005·31 cu. ft. nearly.

42. $16\sqrt{3}$; $\frac{16}{3}\sqrt{2}$. 43. 12 ft.

44. 360 cu. cms. ; 432 sq. cms.

45. 1256·64 sq. inches nearly.

CONTENTS

PART I : FUNCTIONS OF SEVERAL VARIABLES

PART I

FUNCTIONS OF SEVERAL VARIABLES

CHAPTER I

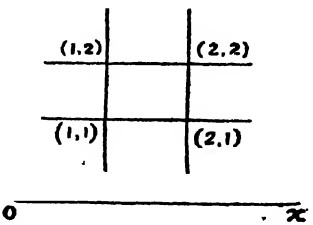
LIMITS AND CONTINUITY

1.1. Functions of Several Variables.

If an expression containing the independent variables $(x_1, x_2 \cdots x_n \cdots)$ be denoted by z and for every set of values of $(x_1, x_2 \cdots x_n \cdots)$ defined in a certain domain R under certain rules, the value of z exists as a real value, then the expression z is called a function of several variables $(x_1, x_2, \cdots x_n \cdots)$ defined in the domain R and we write $z = f(x_1, x_2, \cdots x_n \cdots)$ where z is the dependent variable. It is called a function of two variables, three varibles or more according as the independent variables are two, three or more.

1.2. Domain of the function of two variables.

Let $f(x, y) = \sqrt{(x-1)(2-x)(y-1)(2-y)}$ where the radicands are not negative. Here the two independent variables x and y can assume arbitrary chosen values independent of each other within the ranges $1 \leqslant x \leqslant 2$ and $1 \leqslant y \leqslant 2$.



Geometrically, if we draw a square with vertices at (1, 1), (2, 1), (2, 2) and (1, 2), then the arbitrary set of values of x and y must be chosen from the border or from any point within this square so as to make $f(x, y)$ a single valued function of two variables.

The aggregate of the pair of numbers (x, y) on or within this square is called the domain or region of definition of the function $f(x, y)$ and represented by 'R'. The domain or

region may, however, be a rectangular, a square, a circle or any other finite figure or the whole surface according to the nature of the functions of the two independent variables x and y.

1.3. Domain of function of three variables.

Let $u=\sqrt{(x-1)(2-x)(y-1)(2-y)(z-1)(2-z)}$ where the radicands are not negative. Here the independent variables x, y, z can assume any value within the closed intervals $1 \leqslant x \leqslant 2$; $1 \leqslant y \leqslant 2$ and $1 \leqslant z \leqslant 2$, so as to make u a real valued function of x, y, z.

Geometrically, the ranges of x, y, z can be represented by a cube in three dimensional space. So long x, y, z assume the arbitrary values lying on the surface or within this cube, the value of u has a meaning and we call it a function of x, y, z of three variables defined in a domain R which is a cube here. The domain, however, may be of any solid shape and may be closed or open according to the nature of the function.

1.4. Domain of function of more than three variables.

Let $u=\sqrt{(x_1-1)(2-x_1)(x_2-1)(2-x_2)\cdots(x_n-1)(2-x_n)}$ where the radicands are not negative. Here the ranges of the independent variables $x_1, x_2 \cdots x_n$ are $1 \leqslant x_1 \leqslant 2$; $1 \leqslant x_2 \leqslant 2$ $\cdots 1 \leqslant x_n \leqslant 2$ and for every set of values of $x_1, x_2 \cdots x_n$ within these ranges, the expression u has got some meaning and we call it a function of $(x_1, x_2, \cdots x_n)$ i.e., $u=f(x_1, x_2, \cdots x_n)$.

Geometrically, though we cannot represent these ranges on a solid figure of any shape, yet we agree to say the system of n ordered variables $(x_1, x_2, \cdots x_n)$ correspond to a point in the n dimensional space. The aggregate of all such point in space constitute the domain R_n of the n dimensional space and we say $u=f(x_1, x_2, \cdots x_n)$ over the domain R_n.

1.5. Closed or open domain.

Let $f(x, y)=\sqrt{1-(x^2+y^2)}$ where the radicands are not negative. Here for values of (x, y) on or inside the circle

$x^2+y^2=1$, the function $f(x, y)$ is a single valued function of x and y. So the domain is $x^2+y^2 \leqslant 1$ and therefore closed.

Again if $f(x, y)=\dfrac{1}{\sqrt{1-(x^2+y^2)}}$ where the radicands are not negative then for all arbitrary set of values of (x, y) inside the circle $x^2+y^2=1$, the given function is a singled valued function of x, y. So here the domain is $x^2+y^2<1$ and so open.

1.6. Neighbourhood of a point.

Let $f(x, y)$ be a function of two variables x and y defined in the domain

$$R \equiv (\alpha<x<\beta, \gamma<y<\eta)$$

If (a, b) be a point within this domain, then $R' \equiv (a-\delta<x<a+\delta, b-\delta<y<b+\delta)$ which lies entirely in R is called the neighbourhood of the point (a, b). The domain R may be rectangular but the neighbourhood R' is usually taken as a square. Since $a-\delta<x<a+\delta$ is the same as $0<|x-a|<\delta$, the neighbourhood of (a, b) defines the square domain $0<|x-a|<\delta, 0<|y-b|<\delta$.

Similarly, if $f(x, y)$ be a function of x, y defined in a circular domain R of radius δ and if (a, b) be a point within R then $0<(x-a)^2+(y-b)^2<\delta^2$ defines a circular neighbourhood of (a, b).

1.7. Double or Simultaneous limits of the functions of two variables. Analytical definition :—

A function $f(x, y)$ defined in a certain domain R is said to have a limit l as (x, y) tends to (a, b) which lie entirely in R, when any positive number ϵ having been chosen, as small as we please, there is a positive number δ such that

$$|f(x, y)-l|<\epsilon$$

for all values of (x, y) for which

$$|x-a| \leqslant \delta, |y-b| \leqslant \delta$$

and $\qquad 0<|x-a|+|y-b|$

Thus, it follows that $| f(x, y)-l |$ must be less than ϵ for all points in the square, centre at (a, b) whose sides are parallel to the co-ordinate axes and length of whose side is 2δ, the centre of the square being excluded from the domain.

The square domain may, however, be replaced by a circle with centre at (a, b). The definition of the limit would then be as follows :

A function $f(x, y)$ defined in a certain domain R is said to have a limit l as (x, y) tends to (a, b) which lie entirely in R, if to the arbitrary positive numher ϵ however small, there corresponds a positive number δ depending on ϵ such that

$$| f(x, y)-l | < \epsilon$$

for all values of (x, y) for which

$$0 < \sqrt{(x-a)^2+(y-b)^2} \leq \delta.$$

Note : The limit l is also called the Double limit or Simultaneous limit of $f(x, y)$ as (x, y) tends to (a, b) and is usually written as

$$\underset{(x, y)\to(a, b)}{Lt} f(x, y) \quad \text{or} \quad \underset{\substack{x\to a \\ y\to b}}{Lt} f(x, y)=l.$$

Ex. 1. Verify $\underset{\substack{x\to 0 \\ y\to 0}}{\text{Lt}} f(x, y)=0$

where $f(x, y)=x \sin \frac{1}{x}+y \sin \frac{1}{y},\ x>0,\ y>0.$

for $0<x<\epsilon/2$ **and** $0<y<\epsilon/2$

$$\left| f(x, y)-0 \right| = \left| x \sin \frac{1}{x}+y \sin \frac{1}{y} \right|$$

$$< \epsilon/_2 \left| \sin \frac{2}{\epsilon}+\sin \frac{2}{\epsilon} \right|$$

$$i.e., \quad < \epsilon \left| \sin \frac{2}{\epsilon} \right|$$

$$i.e., \quad < \epsilon \qquad \because \left| \sin \frac{2}{\epsilon} \right| \leq 1.$$

Thus $|f(x, y)-0|<\epsilon$ for all points (x, y) within the square whose sides are along the coordinate axes and of side $\epsilon/_2$.

Hence $\underset{\substack{x\to 0\\ y\to 0}}{Lt} f(x, y)=0.$

Ex. 2. Verify $\underset{\substack{x\to 0\\ y\to 0}}{\mathbf{Lt}} \mathbf{f(x, y)=0}$

where $\mathbf{f(x, y)=(x+y)\sin\frac{1}{x}\sin\frac{1}{y}, x>0, y>0}$

for $\mathbf{0<x<\epsilon/_2}$ **and** $\mathbf{0<y<\epsilon/_2}$

$$\left|f(x, y)-0\right|<\epsilon\left|\sin\frac{2}{\epsilon}\sin\frac{2}{\epsilon}\right|$$

$$i.e., \quad <\epsilon\left|\sin\frac{2}{\epsilon}\right|\left|\sin\frac{2}{\epsilon}\right|$$

$$i.e., \quad <\epsilon \qquad \because \left|\sin\frac{2}{\epsilon}\right|\leqslant 1$$

Thus $|f(x, y)-0|<\epsilon$ for all points (x, y) within the square whose sides are along the co-ordinate axes of side equal to $\epsilon/2$.

Hence $\underset{\substack{x\to 0\\ y\to 0}}{Lt} f(x, y)=0.$

1.8. Analytical definition of the limit of a function of three variables.

A function $f(x, y, z)$ defined in a certain three dimensional domain R is said to tend to the limit l as (x, y, z) tends to (a, b, c) which lie entirely within R, if given any pre-assigned positive number ϵ, however small, we can find a positive number δ depending on ϵ such that

$$|f(x, y, z)-l|<\epsilon$$

for all values of (x, y, z) for which

$$|x-a|\leqslant\delta,\ |y-b|\leqslant\delta,\ |z-c|\leqslant\delta.$$

and $0<|x-a|+|y-b|+|z-c|$

In other words, $|f(x, y, z)-l|<\epsilon$ for all points in the cube, centre at (a, b, c) whose sides are parallel to the co-ordinate axes and of length 2δ. The centre of the square being excluded from the domain.

The cube may, however, be replaced by a sphere and the definition of the limit would then be as follows :

A function $f(x, y, z)$ is said to tend to the limit l as (x, y, z) tends to (a, b, c) if given any pre-assigned positive number ϵ however small, we can find a positive number δ depending on ϵ such that

$$|f(x, y, z)-l|<\epsilon$$

for all values of (x, y, z) for which

$$0<\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}\leq\delta.$$

Ex. 1. **Verify** $\underset{(x, y, z)\to(0, 0, 0)}{\mathrm{Lt}} f(x, y, z)=0$

where $f(x, y, z)=x\sin\frac{1}{x}+y\sin\frac{1}{y}+z\sin\frac{1}{z}$.

For $0<x<\epsilon/3,\ 0<y<\epsilon/3,\ 0<z<\epsilon/3$

$$|f(x, y, z)-0|<\frac{\epsilon}{3}\sin\frac{3}{\epsilon}+\frac{\epsilon}{3}\sin\frac{3}{\epsilon}+\frac{\epsilon}{3}\sin\frac{3}{\epsilon}$$

$$\text{i.e.}<\epsilon\ \sin\frac{3}{\epsilon}$$

$$\text{i.e.}<\epsilon \qquad \because \left|\sin\frac{3}{\epsilon}\right|\leq 1$$

Thus $|f(x, y, z)-0|<\epsilon$ for all value of (x, y, z) lying within a cube whose sides lie along the co-ordinate axes of side $\epsilon/3$.

Hence $\underset{(x, y, z)\to(0, 0, 0)}{Lt} f(x, y, z)=0.$

1.9. Working rule for Evolution of limit of the function of several variables.

Suppose we are to evaluate

$$\underset{\substack{x\to a\\ y\to b}}{Lt} f(x, y$$

If a function $y=\phi(x)$ exists such that $\phi(x)\to b$ as $x\to a$

then $\underset{\substack{x\to a\\ y\to b}}{Lt} f(x,y)=\underset{x\to a}{Lt} f\{x, \phi(x)\}$

If now $\underset{x\to a}{Lt} f\{x, \phi(x)\}$ exists and $=l$, then the double limit is said to exist and we write

$$\underset{\substack{x\to a\\ y\to b}}{Lt.} f(x, y)=l.$$

On the other hand, if two functions $\phi_1(x)$ and $\phi_2(x)$ exist such that each one $\to b$ as $x\to a$ such that

$$\underset{\substack{x\to a\\ y\to b}}{Lt} f(x, y)=\underset{x\to a}{Lt} f\{x, \phi_1(x)\}=l \text{ (say)}$$

$$\underset{\substack{x\to a\\ y\to b}}{Lt} f(x, y)=\underset{x\to a}{Lt} f\{x, \phi_2(x)\}=m \text{ (say)}$$

If $l\neq m$, then the double limit is said to be non-existent.

Ex. 1. Evaluate $\underset{(x, y)\to(0, 0)}{\mathbf{Lt}} \dfrac{2y^2}{\sqrt{x^2+xy}}$.

Let $y=mx$. then as $x\to 0$, $y\to 0$

$$\therefore \underset{(x, y)\to(0, 0)}{Lt} \frac{2y^2}{\sqrt{x^2+xy}}=\underset{x\to 0}{Lt} \frac{2m^2x^2}{\sqrt{x^2+mx^2}}$$

$$=\underset{x\to 0}{Lt} \frac{2m^2x}{\sqrt{1+m}}$$

$$=0 \quad \because \ m \text{ is finite.}$$

Hence $\underset{(x, y)\to(0, 0)}{Lt} \dfrac{2y^2}{\sqrt{x^2+xy}}=0.$

Ex. 2. Evaluate $\underset{(x, y)\to(0, 0)}{\mathbf{Lt}} \dfrac{x^2+y^2}{x^2+xy+y^2}$.

Let $y=mx$ then as $x\to 0$, $y\to 0$

$$\therefore \underset{(x, y)\to(0, 0)}{Lt} \frac{x^2+y^2}{x^2+xy+y^2}=\underset{x\to 0}{Lt} \frac{x^2+m^2x^2}{x^2+mx^2+m^2x^2}$$

$=\underset{x\to 0}{Lt} \dfrac{1+m^2}{1+m+m^2}=\dfrac{1+m^2}{1+m+m^2}$ which is different for different finite values of m.

Hence $\underset{(x,\,y)\to(0,\,0)}{Lt} \dfrac{x^2+y^2}{x^2+xy+y^2}$ does not exist.

1.10. Repeated Limits.

Suppose a function $u=f(x, y)$ is defined in a certain neighbourhood of (a, b) and we are to evaluate

$$\underset{x\to a,\ y\to b}{Lt} f(x, y) \quad \text{or,} \quad \underset{y\to b,\ x\to a}{Lt} f(x, y).$$

1. If $\underset{y\to b}{Lt} f(x, y)$ exist and equal to a function of x, say, $\phi(x)$, then,

$$\underset{x\to a,\ y\to b}{Lt} f(x, y) = \underset{x\to a}{Lt} \phi(x).$$

If now $\underset{x\to a}{Lt} \phi(x)$ also exists and equal to l, a finite or infinite value with a definite sign then, we say that the repeated limit of $f(x, y)$ at (a, b) is l and we write

$$\underset{x\to a,\ y\to b}{Lt} f(x, y) = l$$

where the limit for $y\to b$ is taken first and afterward for $x\to a$.

2. If $\underset{x\to a}{Lt} f(x, y)$ exists and equal to a function of y, say, $\phi(y)$, then

$$\underset{y\to b,\ x\to a}{Lt} f(x, y) = \underset{y\to b}{Lt} \phi(y).$$

If now $\underset{y\to b}{Lt} \phi(y)$ exists and equal to m, a finite or infinite value with a definite sign, then also we say that the repeated limit of $f(x, y)$ at (a, b) is m and we write

$$\underset{y\to b,\ x\to a}{Lt} f(x, y) = m$$

where the limit for $x\to a$ is taken first and afterward for $y\to b$

Note 1. If the double limit $\underset{\substack{x\to a\\y\to b}}{Lt} f(x, y)$ exists and equal to l, a finite or infinite value with a definite sign, then

$$\underset{x\to a,\ y\to b}{Lt} f(x, y)=\underset{y\to b,\ x\to a}{Lt} f(x, y)=l$$

But the converse of the above statement is not always true. Thus $\underset{\substack{x\to a\\y\to b}}{Lt} f(x, y)$ may exist or not although the two repeated limits exist with definite values and equal.

Ex. 1. Let $f(x, y)=x+y+\dfrac{xy}{\sqrt{x^2+y^2}}$, $x>0$, $y>0$ then discuss the existence of double limits and repeated limits of the function at (0, 0).

Let $y=mx$, then as $x\to 0$. $y\to 0$.

$$\therefore \text{ Double } Lt=\underset{\substack{x\to 0\\y\to 0}}{Lt} f(x, y)$$

$$=\underset{x\to 0}{Lt}\left(x+mx+\frac{mx^2}{\sqrt{x^2+m^2x^2}}\right)$$

$$=\underset{x\to 0}{Lt}\left\{1+m+\frac{m}{\sqrt{1+m^2}}\right\}x=0$$

Again $\underset{x\to 0,\ y\to 0}{Lt} f(x, y)=\underset{x\to 0}{Lt}(x)=0$

and $\underset{y\to 0,\ x\to 0}{Lt} f(x, y)=\underset{y\to 0}{Lt}(y)=0$

So double limit of $f(x, y)$ is equal to its repeated limit at (0, 0).

Ex. 2. Let $f(x, y)=\dfrac{x^2y^2}{x^2y^2+(x-y)^4}$; $x>0$, $y>0$. then discuss the existence of the repeated and double limits at (0, 0).

$$\underset{x\to 0,\ y\to 0}{Lt} f(x, y)=\frac{0}{0+(x-0)^4}=\underset{x\to 0}{Lt}(0)=0$$

$$\underset{y\to 0,\ x\to 0}{Lt} f(x, y)=\underset{y\to 0}{Lt}\frac{0}{0+(0-y)^4}=\underset{y\to 0}{Lt}(0)=0$$

So we find that the repeated limits exist and $=0$

For double limit let $y=mx$, then

$$\text{Double } \underset{x\to 0}{Lt} = \underset{x\to 0}{Lt}\ \frac{x^2m^2x^2}{x^2m^2x^2+(x-mx)^4}=\frac{m^2}{m^2+(1+m)^4}$$

which is different for different values of m.

$\therefore$ Double limit does not exist.

Thus although the repeated limits exist and $=0$, the double limit does not exist.

1.11. Important Theorems on Limits

If $\underset{\substack{x\to a\\ y\to b}}{Lt} f(x, y)=l$ and $\underset{\substack{x\to a\\ y\to b}}{Lt} \phi(x, y)=m$

then (a) $\underset{\substack{x\to a\\ y\to b}}{Lt} \{f(x, y)\pm\phi(x, y)\}=l\pm m.$

(b) $\underset{\substack{x\to a\\ y\to b}}{Lt} \{f(x, y).\phi(x, y)\}=lm.$

(c) $\underset{\substack{x\to a\\ y\to b}}{Lt} \left\{\dfrac{f(x, y)}{\phi(x, y)}\right\}=\dfrac{l}{m}$, provided $m\neq 0$

Proofs are exactly the same as in the case of the function of single variable.

1.12. Continuity of a function of two variables.

A function $f(x, y)$ is said to be continuous at a given point (a, b) of its domain of definition if

$$\underset{(x, y)\to(a, b)}{Lt} f(x, y)=f(a, b).$$

Also it is said to be continuous in a domain of definition if it is continuous at every point of the domain.

Analytically : A function $f(x, y)$ is said to be continuous at $x=a$ and $y=b$, if given any pre-assigned positive number ϵ however small, we can find a $\delta>0$ such that

$$|f(x, y)-f(a, b)|<\epsilon$$

for all values of x, y for which

$$|x-a|\leqslant\delta \text{ and } |y-b|\leqslant\delta$$

In other words, $|f(x, y)-f(a, b)|<\epsilon$ for all points in the square, centre at $(a. b)$ whose sides are parallel to the coordinate axes and of length 2δ.

If $f(x, y)$ is continuous at every point of its domain then it is said to be continuous in the domain.

The square domain may, however, be replaced by a circle and the definition would then be as follows :

A function $f(x, y)$ is said to be continuous at (a, b) if to the arbitrary positive number ϵ however small, there corresponds a positive number δ such that

$$|f(x, y)-f(a, b)|<\epsilon$$

for all values of (x, y) for which

$$(x-a)^2+(y-b)^2 \leq \delta^2.$$

1.13. If f(x, y) is a continuous function of two variables at (a, b) then it will be continuous at (a, k) and (k, b) where k is a constant which may be given either to x or y. But the converse of this is not necessarily true.

Consider $f(x, y)=x \sin \frac{1}{x}+y \sin \frac{1}{y}$ where x and $y \neq 0$

$f(x, 0)=x \sin \frac{1}{x}$ when $x \neq 0$; $f(0, y)=y \sin \frac{1}{y}$ when $y \neq 0$ and $f(0, 0)=0$.

Here for $0<x<\epsilon/_2$ and $0<y<\epsilon/_2$

$$|f(x, y)-f(0, 0)| = \left| x \sin \frac{1}{x}+y \sin \frac{1}{y}-0 \right|$$

$$< \left| \frac{\epsilon}{2} \sin \frac{2}{\epsilon}+\frac{\epsilon}{2} \sin \frac{2}{\epsilon} \right|$$

$$\text{i.e. } <\epsilon \left| \sin \frac{2}{\epsilon} \right|$$

$$\text{i.e. } <\epsilon \quad \because \left| \sin \frac{2}{\epsilon} \right| \leq 1$$

So we see that $|f(x, y)-f(0, 0)|<\epsilon$ for every pair of values of x and y within the square whose sides lie along the co-ordinate axes and whose side is of length $\epsilon/_2$

$\therefore$ $f(x, y)$ is continuous at (0, 0).

$$f(x, k)=x \sin \frac{1}{x}+k \sin\frac{1}{k}$$

$$\therefore \underset{x\to 0}{Lt}\, f(x, k)=\underset{x\to 0}{Lt}\, x \sin \frac{1}{x}+k \sin\frac{1}{k}=0+k \sin \frac{1}{k}$$

$$=k \sin \frac{1}{k}$$

Also $\quad f(0, k)=k \sin \frac{1}{k}$

$$\therefore \underset{x\to 0}{Lt}\, f(x, k)=f(0, k)$$

Hence $f(x, y)$ is continious for all values of x including $x=0$. Similarly, it can be shown that $f(x, y)$ is continuous for all values of y including $y=0$

For converse, consider the function

$$f(x, y)=\frac{2xy}{x^2+y^2} \text{ when } x\neq 0, y\neq 0$$

$$=0 \text{ when either } x \text{ or } y \text{ or both}=0.$$

Here $f(x, k)=\dfrac{2xk}{x^2+k^2}$

$$\therefore \underset{x\to 0}{Lt}\, f(x, k)=0$$

and $f(0, k)=0$

$$\therefore \underset{x\to 0}{Lt}\, f(x, k)=f(0, k)$$

$\therefore$ $f(x, k)$ is continuous at $x=0$. So $f(x, y)$ is continuous for all values of x including $x=0$.

Also $f(k, y)=\dfrac{2ky}{k^2+y^2}$

$$\therefore \underset{y\to 0}{Lt}\, f(k, y)=0$$

and $f(k, 0)=0$

$$\therefore \underset{y\to 0}{Lt}\, f(k, y)=f(k, 0)$$

$\therefore$ $f(k, y)$ is continuous at $y=0$. So $f(x, y)$ is continuous for all values of y including $y=0$.

Again consider the continuity of f(x. y) at (0, 0).

Let $y=mx$, then as $x \to 0$, $y \to 0$

$$\therefore \underset{\substack{x\to 0\\ y\to 0}}{Lt}\, f(x, y)=\underset{x\to 0}{Lt}\frac{2x.\,mx}{x^2+m^2x^2}=\frac{2m}{1+m^2}$$ which is different for different values of m. So the limit does not exist.

Hence $f(x, y)$ is not continuous at (0, 0). However, $f(x, y)$ is continuous in any domain excepting the point (0, 0).

1.14. **(a, b) is a point in the region R in which f(x, y) is continuous. If f(a, b)$\neq$0, then f(x, y) will have the same sign as f(a, b) in some neighbourhood of (a, b)** *(C. H. 1963)*

Proof : Since $f(x, y)$ is continuous in (a, b), corresponding to any pre-assigned positive number ϵ, however small, there exists a positive number δ, such that

$$|f(x, y)-f(a, b)| < \epsilon \text{ whenever } 0<|x-a|\leqslant\delta$$
$$\text{and } 0<|y-b|\leqslant\delta.$$

In other words,

$$f(a, b)-\epsilon<f(x, y)<f(a, b)+\epsilon\cdots\cdots\cdots(i)$$
$$\text{whenever } 0<|x-a|\leqslant\delta$$
$$\text{and } 0<|y-b|\leqslant\delta.$$

Case I. Let $f(a, b)>0$

If we now choose ϵ such that $\epsilon<f(a, b)$ then $f(a, b)-\epsilon$ is positive.

Also $f(a, b)+\epsilon$ is always positive.

So from (i) we find that for every point (x, y) in the square domain $R'(a-\delta\leqslant x\leqslant a+\delta\,;\ \ b-\delta\leqslant y\leqslant b+\delta)$, the function $f(x, y)$ lies between two positive quantities and so itself positive.

$\therefore$ $f(x, y)$ has the same sign as $f(a, b)$

Case II Let $f(a, b)<0$

Then $-f(a, b)>0$

If we now choose $\epsilon<-f(a, b)$

Then $f(a, b)+\epsilon<0$

Also $f(a, b)-\epsilon<0 \quad \because f(a, b)<0$

So from (*i*) we find that for every point (x, y) in the square domain R' the function $f(x, y)$ lies between two negative quantities and so itself negative.

$\therefore f(x, y)$ has the same sign as $f(a, b)$. Hence the theorem.

1.15. Some important theorems on continuity of functions of two variables, proofs of which are exactly the same as in the case of a single variable.

1. The sum, difference and product of two functions each continuous at a given point or in a given domain is continuous at the point or in the domain.

Thus if $f(x, y)$ and $\phi(x, y)$ are both continuous at (x_0, y_0) then $f(x, y)\pm\phi(x, y)$ and $f(x, y)\,\phi(x, y)$ are also continuous at (x_0, y_0).

Also if $f(x, y)$ and $\phi(x, y)$ are both continuous in a domain R, then $f(x, y) \pm \phi(x, y)$ and $f(x, y)\,\phi(x, y)$ are also continuous in the domain R.

2. The quotient of two functions both of which are continuous at a point or in a domain is continuous at the point or in the domain except at points where the denominators vanish.

Thus if $f(x, y)$ and $\phi(x, y)$ are both continuous at (x_0, y_0) or in a given domain R and $\phi(x, y)\neq 0$ at (x_0, y_0) or any point in R, then $f(x, y)/\phi(x, y)$ is continuous at (x_0, y_0) or in R except at points in R where $\phi(x, y)$ vanish.

3. A continuous function of one or more continuous function is a continuous function.

In particular let $u=\phi(x, y)$, $v=\psi(x, y)$ be continuous at (x_0, y_0) and let $u_0=\phi(x_0, y_0)$ and $v_0=\psi(x_0, y_0)$.

If now $z=f(u, v)$ be continuous in (u, v) at (u_0, v_0) then

$z=f\{\phi(x, y), \psi(x, y)\}$ is also continuous in x, y at (x_0, y_0).

1.16. Properties of a continuous function of two variables.

(*a*) A function continuous in a closed domain is bounded and attains its bounds at least once in the domain.

(*b*) If $f(x, y)$ is continuous in a closed domain then it must assume at least once every value between its upper and lower bounds.

(*c*) If a function of two variables is continuous at every-point of a closed domain, it is uniformly continuous in the domain.

In other words, corresponding to any pre-assigned positive number ϵ however small, there exists a positive number δ such that

$$| f(x', y') - f(x'', y'') | < \epsilon$$

where (x', y') and (x'', y'') are any two points in the domain for which $\sqrt{(x'-x'')^2+(y'-y'')^2} \leq \delta$.

Illustrative Examples :

Ex. 1. If $f(x, y) = \dfrac{2xy}{x^2+y^2}$, $x^2+y^2 \neq 0$ and $f(0, 0)=0$ show that $f(x, y)$ is not continuous at $(0, 0)$ **(C. H. 1962)**

Let $y=mx$. Then $x \to 0$ as $y \to 0$

$$\therefore \underset{(x, y)\to(0, 0)}{Lt} f(x, y) = \underset{x\to 0}{Lt} \frac{2mx^2}{x^2+m^2x^2} = \frac{2m}{1+m^2}$$ which is different for different values of m.

$$\therefore \underset{(x, y)\to(0, 0)}{Lt} f(x, y) \text{ does not exist}$$

but $f(0, 0)=$exists which is zero

$\therefore$ Lt $f(x, y) \neq f(0, 0)$ at $(0, 0)$

Hence $f(x, y)$ is not continuous at $(0, 0)$

Ex. 2. If $f(x, y) = \dfrac{x^2}{x^2+y^2-x}$, $x^2+y^2 \neq 0$ and $f(0, 0)=0$, examine whether $f(x, y)$ is continuous at $(0, 0)$ **(C. H. 1962)**

Suppose we proceed along the curve $y^2 - x = mx^2$. Then as $x \to 0$, $y \to 0$.

$\therefore \underset{(x,\,y)\to(0,\,0)}{Lt} f(x, y)=\underset{x\to 0}{Lt} \dfrac{x^2}{x^2+mx^2}=\dfrac{1}{1+m}$ which is different for different values of m

$\therefore \underset{(x,\,y)\to(0,\,0)}{Lt} f(x, y)$ does not exist

But $f(x, y)$ is defined at $(0, 0)$

$\therefore \underset{(x,\,y)\to(0,\,0)}{Lt} f(x, y)\neq f(0, 0)$

Hence $f(x, y)$ is not continuous at $(0, 0)$.

Ex. 3. Examine whether

$$\mathbf{f(x, y)}=\begin{cases} \mathbf{xy} & \textbf{if } |\mathbf{x}| \geqslant |\mathbf{y}| \\ -\mathbf{xy} & \textbf{if } |\mathbf{x}| < |\mathbf{y}| \end{cases}$$

is continuous at the origin. **(C. H. (old) 1963)**

Let $y=mx$, then $y\to 0$ as $x\to 0$

$\therefore \underset{(x,\,y)\to(0,\,0)}{Lt} f(x, y)$

$=\underset{x\to 0}{Lt}\, mx^2$ if $|x| \geqslant |y|$

$=0$

and $\underset{(x,\,y)\to(0,\,0)}{Lt} f(x, y)=\underset{x\to 0}{Lt} -(mx^2)$ if $|x| < |y|$.

$=0$

Also $f(0, 0)=0$

Thus $\underset{(x,\,y)\to(0,\,0)}{Lt} f(x, y)=f(0, 0)$

Hence $f(x, y)$ is continuous at $(0, 0)$.

Ex. 4. Examine the continuity of the function

$\mathbf{f(x, y)}=\dfrac{\mathbf{x^3+y^3}}{\mathbf{x^2+y^2}}$, $\mathbf{x^2+y^2\neq 0}$ **(C. H. 1961)**

Let $y=mx$, then $y\to 0$ as $x\to 0$

$$\underset{(x,\,y)\to(0,\,0)}{Lt} f(x, y)=\underset{x\to 0}{Lt} \frac{x^3+m^3x^3}{x^2+m^2x^2}$$

$$=\underset{x\to 0}{Lt} \frac{1+m^3}{1+m^2}\cdot x=0.$$

but $f(0, 0)$ is not defined in the question.

$\therefore \quad \underset{(x,\, y)\to(0,\, 0)}{Lt} f(x, y) \neq f(0, 0)$

$\therefore \quad f(x, y)$ is not continuous at $(0, 0)$

Again quotient of $\dfrac{x^3+y^3}{x^2+y^2}$ is a polynomial and so it is continuous for all values of x excepting at $x=0,\ y=0$.

Hence $f(x, y)$ is continuous everywhere excepting at $(0, 0)$.

Exercise 1

1. Show that $\underset{(x,\, y)\to(0,\, 0)}{Lt} \dfrac{x^2 y}{x^4+y^4}$ does not exist.

2. Show that $\underset{(x,\, y)\to(0,\, 0)}{Lt} \dfrac{(xy+y^2)+(x+y)^3}{y-(x+y)^2}$ does not exist.

3. Let $f(x, y)=\dfrac{x^2y^2}{x^2y^2+(x-y)^2}$, show that the two repeated limits of $f(x, y)$ exist and are equal.

Also show that $\underset{(x,\, y)\to(0,\, 0)}{Lt} f(x, y)$ does not exist.

4. Show that a functio..

$f(x, y)=xy\,\dfrac{x^2-y^2}{x^2+y^2}$, $f(x, y)\neq(0, 0)$

and $f(0, 0)=0$ is continuous at $(0, 0)$.

5. Let a function of two variables in x and y is defined as

$f(x, y)=e^{-\,|\,x-y\,|\,/(x^2-2xy+y^2)}$, when $(x, y)\neq(x, x)$

and $f(x, x)=0$, show that $f(x, y)$ is continuous at $(0, 0)$.

6. Show that a function defined as

$f(x, y)=\dfrac{x^3}{x^2+y^2-2x}$ when $(x, y)\neq(0, 0)$ and $f(0, 0)=0$

is not continuous at $(0, 0)$.

7. If $\underset{(x,\, y)\to(a,\, b)}{Lt} f(x, y)=l$, then show that

$$\underset{x\to a}{Lt} f(x, b)=\underset{y\to b}{Lt} f(a, y)=l.$$

8. Let $f(x, y)=(x+y)\sin\dfrac{1}{x}\sin\dfrac{1}{y}$, $x>0,\ y>0$.

Discuss the existence of the repeated and double limits at $(0, 0)$.

9. Show that the function $f(x, y)=y^2\sin\dfrac{1}{x}$, $x\neq0$

and $f(0, 0)=0$ is continuous at the origin.

CHAPTER II

PARTIAL DERIVATIVES

2.1. Let $z=f(x, y)$

Then $\underset{h\to 0}{Lt} \dfrac{f(a+h, b)-f(a, b)}{h}$ if exist is called the partial derivative of $f(x, y)$ with respect to x at (a, b) and it is denoted by $\left(\dfrac{\partial z}{\partial x}\right)_{(a, b)}$ or $f_x(a, b)$.

Thus $f_x(a, b)=\underset{h\to 0}{Lt} \dfrac{f(a+h, b)-f(a, b)}{h}$

If, however, $f(x, y)$ possesses partial derivative with respect to x, at every point of its domain of definition, then at any point (x, y) of the domain

$$f_x(x, y)=\underset{h\to 0}{Lt} \frac{f(x+h, y)-f(x, y)}{h}$$

Again $\dfrac{f(a, b+k)-f(a, b)}{k}$ if exist is called the partial derivative of $f(x, y)$ with respect to y at (a, b) and it is denoted by $\left(\dfrac{\partial z}{\partial y}\right)_{(a,b)}$ or $f_y(a, b)$.

Also if $f(x, y)$ possesses partial derivative with respect to y at every point of its domain of definition, then at any point (x, y) of its domain

$$f_y(x, y)=\underset{k\to 0}{Lt} \frac{f(x, y+k)-f(x, y)}{k}.$$

Illustrations :

Ex. 1. If $f(x, y)=0$ when $x=0$ or $y=0$ and $f(x, y)=1$ when $x\neq 0$, $y\neq 0$, find f_x and f_y at $(0, 0)$.

$$f_x(0, 0)=\underset{h\to 0}{Lt} \frac{f(0+h, 0)-f(0, 0)}{h}=\underset{h\to 0}{Lt} \frac{0-0}{h}=0.$$

$$f_y(0, 0)=\underset{k\to 0}{Lt} \frac{f(0, 0+k)-f(0, 0)}{k}=\underset{k\to 0}{Lt} \frac{0-0}{k}=0.$$

Hence both f_x and f_y exist at $(0, 0)$ and $=0$.

2.2. Partial Derivatives and Continuity.

Let $z=f(x, y)$

Then the partial derivatives of z with respect to x and y at (a, b) are respectively

$$\left(\frac{\partial z}{\partial x}\right)_{(a, b)}=\underset{h\to 0}{Lt}\ \frac{f(a+h, b)-f(a, b)}{h}$$

and $$\left(\frac{\partial z}{\partial y}\right)_{(a, b)}=\underset{k\to 0}{Lt}\ \frac{f(a, b+k)-f(a, b)}{k}$$

Thus in forming partial derivatives we vary x along the line $x=a$ in the first case and y along the line $y=b$ in the second case. So $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ exist at (a, b) only when the function $f(x, y)$ exist for points on the lines $x=a$ and $y=b$. For a point outside these two lines plays no roll on partial derivatives. But for continuity of $f(x, y)$ at (a, b) we have to consider the existence and values of the function not only on the lines $x=a$ and $x=b$ but at every point on the neighbourhood of (a, b). *Thus a function may possess partial derivatives without being continuous at the point.* If, however, the partial derivatives are bounded in the domain of definition of the function, then the function must be continuous in that domain.

Theorem :

If a function f(x, y) defined in a domain R have partial derivatives f_x and f_y in R and further if they are bounded in R, then f(x, y) is continuous in R. (C. H. 1962)

Let us write

$$f(x+h, y+k)-f(x, y)=\{f(x+h, y+k)-f(x, y+k)\}$$
$$+\{f(x, y+k)-f(x, y)\} \quad \ldots \quad (1)$$

Since f_x and f_y are bounded in R, they are finite in R

$\therefore$ $f(x, y)$ is derivable in R with respect to x and y.

Hence applying mean value theorem in (1)

$$f(x+h, y+k)-f(x, y)=hf_x(x+\theta h, y+k)$$
$$+hf_y(x, y+\theta' k) \quad \ldots \quad (2)$$

for $0<\theta<1$
and $0<\theta'<1.$

Again if M be the greatest of the upper bounds of f_x and f_y

Then we must have

$$|f_x| \leqslant M \text{ and } |f_y| \leqslant M$$

So from (2)

$$|f(x+h, y+k)-f(x, y)| \leqslant |hf_x(x+\theta h, y+k)| + |hf_y(x, y+\theta' k)|$$

i.e., $\leqslant M\{|h|+|k|\}$

i.e., $< M.\frac{\epsilon}{M}$

$\because$ we can choose $|h|+|k|<\frac{\epsilon}{M}$

i.e., $<\epsilon$

So it follows that $f(x, y)$ is continuous at (x, y) in the domain R.

Cor. If f_x and f_y are continuous at (x, y) in R, then f(x, y) is continuous in R.

Since f_x and f_y are continuous at (x, y) in R, they must be bounded in R and so $f(x, y)$ is continuous in R (by the last theorem).

Illustrations :

Ex. 1. Let $f(x, y)=\frac{xy}{x^2+y^2}, (x, y)\neq 0$

$=0$, **otherwise**

Prove that $f_x(0, 0)$ and $f_y(0, 0)$ both exist but $f(x, y)$ is discontinuous at (0, 0). (C. H. 1966)

$$f_x(0, 0)=\underset{h\to 0}{Lt}\frac{f(h, 0)-f(0, 0)}{h}$$

$$=\underset{h\to 0}{Lt}\frac{0-0}{h}=0.$$

$$f_y(0, 0)=\underset{k\to 0}{Lt}\ \frac{f(0, k)-f(0, 0)}{k}$$

$$=\underset{k\to 0}{Lt}\ \frac{0-0}{k}=0.$$

Thus both f_x and f_y exist at (0, 0) and their value$=0$.

Again let us suppose that (x, y) approaches (0, 0) along the line $y=mx$.

Then $\underset{(x, y)\to(0, 0)}{Lt} f(x, y)=\underset{(x, y)\to(0, 0)}{Lt} \frac{mx^2}{x^2+m^2x^2}$

$=\frac{m}{1+m^2}$, which is different for different values of m.

Hence $f(x, y)$ is discontinuous at (0, 0).

Ex. 2. If f(x, y)=0 when either x=0 or y=0 and f(x, y) =1 when x≠0, y≠0.

Show that f(x, y) is discontinuous at (0, 0) but the partial derivatives exist at (0, 0).

Since $|f(x, y)-f(0, 0)| = |1-0| =1 \nless \epsilon$, $f(x, y)$ is not continuous at (0, 0).

Again $f_x(0, 0)=\underset{h\to 0}{Lt}\ \frac{f(0+h, 0)-f(0, 0)}{h}=\underset{h\to 0}{Lt}\left(\frac{0}{h}\right)=0$

$$f_y(0, 0)=\underset{k\to 0}{Lt}\ \frac{f(0, 0+k)-f(0, 0)}{k}=\underset{k\to 0}{Lt}\left(\frac{0}{k}\right)=0$$

Thus partial derivatives of $f(x, y)$ with respect to x and y exist at (0, 0) although $f(x, y)$ is discontinuous at (0, 0).

Ex. 3. Prove that the function f(x, y) defined by

$f(x, y)=\frac{2xy}{x^2+y^2}$, $x^2+y^2\neq 0$ and $f(0, 0)=0$ has partial derivatives everywhere. Is it continuous every where ?

(C. H. 1962)

$$f_x=\underset{h\to 0}{Lt}\ \frac{f(x+h, y)-f(x, y)}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{1}{h}\left[\frac{2(x+h)y}{(x+h)^2+y^2}-\frac{2xy}{x^2+y^2}\right]$$

$$=\underset{h\to 0}{Lt}\ \frac{2}{h}\ \frac{\{(xy+hy)(x^2+y^2)-xy(x^2+y^2+h^2+2xh)\}}{(x^2+y^2)(x^2+y^2+h^2+2xh)}$$

$$=\underset{h\to 0}{Lt}\ \frac{2(y^3-x^2y-xyh)}{(x^2+y^2)(x^2+y^2+h^2+2xh)}=\frac{2y(y^2-x^2)}{(x^2+y^2)^2}$$

which exists for all values of x other than $x=0,\ y=0$.

Similarly, it can be shown that f_y exists for all values of y other than $x=0,\ y=0$.

Again $f_x(0, 0)=\underset{h\to 0}{Lt}\ \dfrac{f(h, 0)-f(0, 0)}{h}$

$$=\underset{h\to 0}{Lt}\ \frac{0-0}{h}=0.$$

Similarly, $f_y(0, 0)=0$.

Thus we find that $f(x, y)$ has derivatives every where.

For continuity of the function see Art 1·13. page 12.

2.3. Differentiability of a function f(x, y).

A function $u=f(x, y)$ is said to be differentiable at a point (x, y) of its domain of definition, if the increment δu of u, corresponding to any arbitrary assigned increment δx and δy of x and y can be expressed in the form

$$\delta u=f(x+h, y+k)-f(x, y)$$

$$=Ah+Bk+h\phi(h, k)+k\psi(h, k) \quad \cdots \quad \cdots \quad (1)$$

where A and B are constant independent of h and k; $\phi(h, k)$ and $\psi(h, k)$ are functions of (h, k) such that

$$\underset{(h, k)\to(0, 0)}{Lt}\ \phi(h, k)=0$$

and $$\underset{(h, k)\to(0, 0)}{Lt}\ \psi(h, k)=0.$$

The part of δu in (1) which is linear in h and k *i.e.*, the part $Ah+Bk$ is called the total differential of u and is denoted by du.

So we write $du=Ah+Bk$

$$=A\delta x+B\delta y \quad \cdots \quad \cdots \quad (2)$$

$\because$ h and k are increment of $x+y$

$$h=\delta x,\ k=\delta y$$

Now From (1)

$$\delta u=A\delta x+B\delta y+\delta x\phi(\delta x, \delta y)+\delta y\psi(\delta x\ \delta y) \cdots \quad \cdots \quad (3)$$

Suppose we now regard y as fixed and x as variable, then $\delta y=0$

$\therefore$ from (3)

$$\delta u=A\delta x+\delta x.\phi(\delta x, 0)$$

or, $\dfrac{\delta u}{\delta x}=A+\phi(\delta x, 0)$.

or. $\underset{\delta x\to 0}{Lt}\dfrac{\delta u}{\delta x}=A$

or, $\dfrac{\partial u}{\partial x}=A \quad \because \quad y$ is regarded as a constant.

Similarly, if we regard x as fixed and y as variable then $\delta x=0$ and we shall get

$\dfrac{\partial u}{\partial y}=B \quad \because \quad x$ is regarded as a constant

Hence from (2) we get

$$du=\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y \qquad \cdots \qquad \cdots \qquad (4)$$

But $\dfrac{\partial u}{\partial x}\delta x$ and $\dfrac{\partial u}{\partial y}\delta y$ are called the partial differentials of u with respect to x and y respectively. So it follows that *the total differential of u is the sum of its partial differentials.*

If now we put $u=f(x, y)=x$

Then $\dfrac{\partial u}{\partial x}=1$ and $\dfrac{\partial u}{\partial y}=0$

$\therefore$ From (4) we get $dx=\delta x$.

Similarly, if we put $u=f(x, y)=y$, we shall get $dy=\delta y$.

Hence (4) can be written as

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy.$$

Similarly, for n number of variables x, y, z $\cdots$

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz+\cdots$$

From this it follows that a function is differentiable at a point if it possess a total differential at that point.

Theorem :

The necessary condition for a function $f(x, y)$ to be differentiable at a point (a, b) of its domain is that $f(x, y)$ should be continuous at that point.

Since $u=f(x, y)$ is differentiable at (a, b) the increment of u can be expressed in the form

$$\delta u= Ah+Bk+h\phi(h, k)+k\psi(h, k) \quad \cdots \quad (1)$$

where A and B are constants independent of h and k; $\phi(h, k)$ and $\psi(h, k)$ are both functions of h and k and both tend to zero as $(h, k)\to(0, 0)$.

But $\delta u=f(a+h, b+k)-f(a, b)$

$\therefore$ from (1)

$$f(a+h, b+k)-f(a, b)=Ah+Bk+h\phi(h, k)+k\psi(hk) \quad \cdots \quad (2)$$

The linear portion $Ah+Bk$ is the differential of $f(x, y)$ at (a, b)

$\therefore$ $Ah+Bk=d.f(a, b)=0$ in the limit.

Hence from (2)

$$\underset{(h, k)\to(0, 0)}{Lt} \{f(a+h, b+k)-f(a, b)\}=0$$

or, $$\underset{(h, k)\to (0, 0)}{Lt} f(a+h, b+k)=f(a, b)$$

$\therefore$ $f(x, y)$ is continuous at (a, b)

The Converse of the above theoıem is not always true, for there are functions which are continuous at some points of their domain but are not differentiable at those points.

Illustrations :

Let $f(x, y)=\dfrac{xy}{\sqrt{x^2+y^2}}$, $(x, y)\neq(0, 0)$ and $f(0, 0)=0$,

Show that $f(x, y)$ is continuous and possesses partial derivatives at $(0, 0)$ but is not differentiable at that point.

Suppose (x, y) approaches $(0, 0)$ along the line $y=mx$ then $x\to0$ as $y\to0$

$$\therefore \quad \underset{(x,\,y)\to(0,\,0)}{Lt} f(x,\,y)=\underset{x\to 0}{Lt}\frac{x.mx}{\sqrt{x^2+m^2x^2}}$$

$$=\underset{x\to 0}{Lt}\frac{mx}{\sqrt{1+m^2}}=0$$

and given $f(0,\,0)=0$

$$\therefore \quad \underset{(x,\,y)\to(0,\,0)}{Lt} f(x,\,y)=f(0,\,0)$$

$\therefore$ $f(x,\,y)$ is continuous at $(0,\,0)$.

Again $\quad f_x(0,\,0)=\underset{h\to 0}{Lt}\dfrac{f(0+h,\,0)-f(0,\,0)}{h}.$

$$=\underset{h\to 0}{Lt}\frac{\dfrac{h\times 0}{\sqrt{h^2+0}}-0}{h}=\underset{h\to 0}{Lt}\frac{0-0}{h}=0.$$

$$f_y(0,0)=\underset{k\to 0}{Lt}\frac{f(0,\,0+k)-f(0,\,0)}{k}.$$

$$=\underset{k\to 0}{Lt}\frac{\dfrac{0\times k}{\sqrt{0+k^2}}-0}{k}=\underset{k\to 0}{Lt}\frac{0-0}{k}=0.$$

Now the condition for differentiability of the function at $(0,\,0)$ is the validity of the relation

$$f(0+h,\,0+k)-f(0,\,0)=Ah+Bk+h\phi(h,\,k)+k\psi(h,\,k) \quad \cdots (1)$$

Where $\quad A=f_x(0,\,0)=0$

$\qquad\quad B=f_y(0,\,0)=0$

And $\quad \underset{(h,\,k)\to(0,\,0)}{Lt}\phi(h,\,k)=0,\quad \underset{(h,\,k)\to(0,\,0)}{Lt}\psi(h,\,k)=0.$

$\therefore$ from (1)

$$f(h,\,k)-f(0,\,0)=h\phi(h,\,k)+k\psi(h,\,k)$$

$$=h\left\{\phi(h,\,k)+\frac{k}{h}\psi(h,\,k)\right\}$$

or, $\quad \dfrac{k}{\sqrt{h^2+k^2}}=\phi(h,\,k)+\dfrac{k}{h}\psi(h,\,k).$

or, $\quad \underset{(h,\,k)\to(0,\,0)}{Lt}\dfrac{k}{\sqrt{h^2+k^2}}$

$$=\underset{(h,\,k)\to(0,\,0)}{Lt}\left\{\phi(h,\,k)+\frac{k}{h}\psi(h,\,k)\right\}=0. \quad \cdots (2)$$

To evaluate the *L.H.S.* let us suppose that (h, k) approaches (0, 0) along the line $k=mh$

Then $L.H.S = \underset{h\to 0}{Lt} \dfrac{mh}{\sqrt{h^2+m^2h^2}} = \dfrac{m}{\sqrt{1+m^2}}$

So from (2) we get

$$\frac{m}{\sqrt{1+m^2}}=0 \text{ which is impossible.}$$

$\because$ m is a non zero positive quantity, $\dfrac{m}{\sqrt{1+m^2}}$ can never be zero.

So the relation (1) is not valid and hence $f(x, y)$ is not differentiable at (0, 0).

Theorem :

The necessary condition for a function f(x, y) to be differentiable at (x, y) is that the function should possess partial derivatives f_x and f_y at that point.

Let $u=f(x, y)$ be a function of x, y defined in the domain R.

Then the function $f(x, y)$, is said to be differentiable at (x, y) if it possess total differential du at that point such that

$$du=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}. dy$$

So for existence of du, the existence of both $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are essential.

Hence if $u=f(x, y)$ is differentiable at (x, y), then it must possess partial derivatives f_x and f_y at that point.

But the converse of this theorem is not true for there are functions which have partial derivatives but are not differentiable at that point.

Illustrations :

For the function $f(x, y)=(|xy|)^{\frac{1}{2}}$ both f_x and f_y exist at (0, 0) but is not differentiable at (0, 0).

$$f_x(0, 0)=\underset{h\to 0}{Lt}\frac{f(h, 0)-f(0, 0)}{h}=\underset{h\to 0}{Lt}\frac{0-0}{h}=0.$$

$$f_y(0, 0)=\underset{k\to 0}{Lt}\frac{f(0, k)-f(0, 0)}{k}=\underset{k\to 0}{Lt}\frac{0-0}{k}=0$$

Thus f_x and f_y both exist at (0, 0).

Again for differentiable at (0, 0) we have the relation

$$f(0+h, 0+k)-f(0, 0)=Ah+Bk+h\phi(h, k)+k\psi(h, k)\cdots\cdots(1)$$

where $A=f_x(0, 0)=0$

$B=f_y(0, 0)=0$

and $\underset{(h, k)\to(0, 0)}{Lt}\phi(h, k)=0, \quad \underset{(h, k)\to(0, 0)}{Lt}\psi(h, k)=0.$

$\therefore$ from (1) $\quad f(h, k)-f(0, 0)=h\phi(h, k)+k\psi(h, k)$

$$\text{or,}\quad \underset{(h, k)\to(0, 0)}{Lt}\left\{f(h, k)-f(0, 0)\right\}$$

$$=\underset{(h, k)\to(0, 0)}{Lt}\left\{h\phi(h, k)+k\psi(h, k)\right\}$$

$$\text{or,}\quad \underset{(h, k)\to(0, 0)}{Lt}\left\{|h.k|\right\}^{\frac{1}{2}}=\underset{(h, k)\to(0, 0)}{Lt}\left\{h\phi(h, k)+k\psi(h, k)\right\}$$

$$\text{or,}\quad \underset{(h, k)\to(0, 0)}{Lt}\frac{\{|h, k|\}^{\frac{1}{2}}}{h}=\underset{(h, k)\to(0, 0)}{Lt}\left\{\phi(h, k)+\frac{k}{h}\psi(h, k)\right\}$$

$$=0.\cdots\ \cdots\ \cdots(2)$$

Now $L.H.S.=\underset{h\to 0}{Lt}\dfrac{\{h.\,mh\}^{\frac{1}{2}}}{h}=\sqrt{m}$ putting $k=mh$.

$\therefore$ from (2) $\sqrt{m}=0$ which is absurd as $\sqrt{m}$ may have any value not necessarily zero.

$\therefore$ The relation (1) is not true and hence $f(x, y)$ is not differentiable at (0, 0).

Theorem :

A sufficient condition for the differentiability of a function $u=f(x, y)$, at a given point (a, b) of its domain, is that (i) $f_x(a, b)$ exists and (ii) $f_y(x, y)$ is continuous at (a, b).

(C. H. 1968)

Since $f_y(x, y)$ is continuous at (a, b), $f_y(x, y)$ exists in a certain neighbourhood of (a, b). Let $(a+h, b+k)$ be any point of this neighbourhood. Then for values within this neighbourhood, we can write :

$$f(a+h, b+k)-f(a, b)=f(a+h, b+k)-f(a+h, b)+ f(a+h,b)-f(a, b) \cdots\cdots(1)$$

The function $f(a+h, y)$ of y is derivable with respect to y in $(b, b+k)$. So by Mean Value Theorem

$$f(a+h, b+k)-f(a+h, b)=kf_y(a+h, b+\theta k) \cdots\cdots (2)$$

where $0<\theta<1$.

If we construct a function $\psi(h, k)$ such that

$$\psi(h, k)=f_y(a+h, b+\theta k)-f_y(a, b) \cdots\cdots(3)$$

Then $\underset{(h, k)\to(0, 0)}{Lt} \psi(h, k)= \underset{(h, k)\to(0, 0)}{Lt} \left\{f_y(a+h, b+\theta k) -f_y(a, b)\right\}$

$$=f_y(a, b)-f_y(a, b)$$

$$=0 \quad \because f_y \text{ is continuous at } (a, b)$$

Also since f_x exists at (a, b)

$$\underset{h\to0}{Lt} \frac{f(a+h, b)-f(a, b)}{h}=f_x(a, b)$$

So we can write

$$\frac{f(a+h, b)-f(a, b)}{h}-f_x(a, b)=\phi(h) \cdots\cdots(4)$$

where $\phi(h)\to0$ as $h\to0$.

From (2) and (3)

$$f(a+h, b+k)-f(a+h, b)=k\psi(h, k)+kf_y(a, b) \cdots\cdots(5)$$

From (4)

$$f(a+h, b)-f(a, b)=hf_x(a, b)+h\phi(h) \cdots\cdots(6)$$

Putting (5) and (6) in (1) we get

$$f(a+h, b+k)-f(a, b)=hf_x(a, b)+kf_y(a, b)+h\phi(h)+k\psi(h, k)$$

which is the condition for differentiability of $f(x, y)$ at (a, b). Hence, the theorem.

Note : If $f_y(a, b)$ exist and $f_x(x, y)$ is continuous at (a, b), then also $f(x, y)$ is differentiable at (a, b). Thus the sufficient condition of differentiability at (a, b) is that only one of the partial derivatives exists at (a, b) and the other is continuous at (a, b). *But there are functions which are differentiable although no partial derivative is continuous.*

This is illustrated in the following example :—

Let $\mathbf{f(x, y)=x^2 \sin \frac{1}{x}+y^2 \sin \frac{1}{y}, x\neq 0, y\neq 0}$

$\mathbf{f(x, 0)=x^2 \sin \frac{1}{x}, x\neq 0}$

$\mathbf{f(0, y)=y^2 \sin \frac{1}{y}, y\neq 0}$

and $\mathbf{f(0, 0)=0.}$

Show that $\mathbf{f_x}$ **and** $\mathbf{f_y}$ **are discontinuous at the origin but** $\mathbf{f(x, y)}$ **is differentiable at the origin.**

$\because \quad f(x, y)=x^2 \sin \frac{1}{x}+y^2 \sin \frac{1}{y}, \ x\neq 0, \ y\neq 0$

$f_x(x, y)=2x \sin \frac{1}{x}-\cos \frac{1}{x}, \ x\neq 0$ and y remaining constant.

and $f_y(x, y)=2y \sin \frac{1}{y}-\cos \frac{1}{y}, \ y\neq 0$ and x remaining constant.

But $f_x(0, y)=\underset{h\to 0}{Lt} \dfrac{f(0+h, y)-f(0, y)}{h}$

$$=\underset{h\to 0}{Lt} \frac{h^2 \sin \frac{1}{h}+y^2 \sin \frac{1}{y}-y^2 \sin \frac{1}{y}}{h}$$

$$=\underset{h\to 0}{Lt}\ h \sin \frac{1}{h}=0.$$

and $f_y(x, 0)=\underset{k\to 0}{Lt}\ \dfrac{f(x, 0+k)-f(x, 0)}{k}$

$$=\underset{k\to 0}{Lt}\ \frac{x^2 \sin \frac{1}{x}+k^2 \sin \frac{1}{k}-x^2 \sin \frac{1}{x}}{k}$$

$$=\underset{k\to 0}{Lt}\ k \sin \frac{1}{k}=0$$

$\therefore$ f_x and f_y is not continuous at (0, 0)

Now if $f(x, y)$ is differentiable at (0, 0), then we must have

$$f(0+h, 0+k)-f(0, 0)=h^2 \sin \frac{1}{h}+k^2 \sin \frac{1}{k}$$
$$=Ah+Bk+h\phi(h, k)+k\psi(h, k)\cdots \qquad \cdots \quad (1)$$

where $A=f_x(0, 0)=\underset{h\to 0}{Lt}\ \dfrac{f(h, 0)-f(0, 0)}{h}$

$$=\underset{h\to 0}{Lt}\frac{\left(h^2 \sin \frac{1}{h}-0\right)}{h}=0$$

$B=f_y(0, 0)=\underset{k\to 0}{Lt}\ \dfrac{f(0, k)-f(0, 0)}{k}$

$$=\underset{k\to 0}{Lt}\frac{k^2 \sin \frac{1}{k}-0}{k}=0$$

and $\underset{(h, k)\to(0, 0)}{Lt}\ \phi(h, k)=\underset{(h, k)\to(0, 0)}{Lt}\ \psi(h, k)=0\cdots(2)$

So from (1) we find that

$$h^2 \sin\frac{1}{h}+k^2 \sin\frac{1}{k}=0.\ h+0.\ k+h\left(h \sin\frac{1}{h}\right)+k\left(k \sin\frac{1}{k}\right)\cdots \quad (3)$$

$\because$ $\phi(h, k)=h \sin \frac{1}{h}$ and $\psi(h, k)=k \sin\frac{1}{k}$

both being zero as $(h, k)-(0, 0)$

This shows that (3) is identically satisfied.

Hence, the give function is differentiable at the origin although their partial derivatives do not exist at (0, 0).

2.4. Calculation of small errors by differentials.

When the values of x and y are obtained by any measurement are subject to small errors dx and dy, the corresponding error dz of z can be calculated approximately by the formula.

$$dz=\frac{\partial f}{\partial x}\,dx+\frac{\partial f}{\partial y}\,dy. \qquad (1)$$

For example, consider the volume of a cone of height h and radius of the base r as

$$V=\tfrac{1}{3}\pi r^2 h.$$

Then $\dfrac{\partial V}{\partial r}=\tfrac{2}{3}\pi rh$ and $\dfrac{\partial V}{\partial h}=\tfrac{1}{3}\pi r^2$.

using formula (1)

$$dV=(\tfrac{2}{3}\pi rh)dr+(\tfrac{1}{3}\pi r^2)dh$$

or, $$\frac{dV}{V}=\frac{\tfrac{2}{3}\pi rh}{V}dr+\frac{\tfrac{1}{3}\pi r^2}{V}dh$$

$$=\frac{2}{r}dr+\frac{1}{h}dh. \qquad \cdots \qquad (2)$$

So in measuring r and h, if there happens to be an error of dr and dh respectively, then by (2) it is possible to calculate $\dfrac{dV}{V}$ which measures the relative error in volume of the cone.

In particular if r and h are measured as 4 and 8 inches with the possible error of ·04 and ·08 inches then

$$\frac{dV}{V}=\frac{2}{4}\times\cdot 04+\frac{1}{8}\times\cdot 08=\cdot 03$$

$\therefore$ Percentage error $=\dfrac{100dV}{V}=3\%$.

Ex. 1. The side a of the triangle ABC is calculated by measuring the sides b, c and the angle A. If the corresponding error be db, dc and dA, show that the error in the calculated value of a is

$$\mathbf{da=(b\sin C)\,dA+(\cos C)db+(\cos B)dc.}$$

We have $a^2=b^2+c^2-2bc\cos A$

$\therefore\quad 2a\,da=2b\,db+2c\,dc-2(c\cos A\,db+b\cos A\,dc$
$-bc\sin A\,dA)$

or, $da=(b-c\cos A)\dfrac{db}{a}+(c-b\cos A)\dfrac{dc}{a}+bc\dfrac{\sin A}{a}\,dA$

[$\because$ $b=c\cos A+a\cos C,\ c=a\cos B+b\cos A$]

$$=a\cos C\,\frac{db}{a}+a\cos B\,\frac{dc}{a}+bc\frac{\sin A}{a}\,dA$$

$$=(\cos C)db+(\cos B)dc+bc.\frac{\sin C}{c}\,dA$$

$$=(\cos C)db+(\cos B)dc+(b\sin C)\,dA.$$

Ex. 2. The side a and the opposite angle A of a triangle ABC remains constant. Show that when the other sides and angles are slightly varied, then. (C. H. 1969)

$$\frac{db}{\cos B}+\frac{dc}{\cos C}=0$$

We have

$b\cos C+c\cos B=a$

$\therefore\quad db\cos C-b\sin C\,dC+dc\cos B-c\sin B\,dB=0.$

or, $db\cos C+dc\cos B-\{(b\sin C)dC+(c\sin B)dB\}=0$

or, $db\cos C+dc\cos B-c\sin B(dC+dB)=0$

$\because\quad \dfrac{\sin B}{b}=\dfrac{\sin C}{c}$, i.e., $b\sin C=c\sin B.$

or, $db\cos C+dc\cos B-c\sin B\,d(C+B)=0$

or, $db\cos C+dc\cos B-c\sin B\,d(\pi-A)=0.$

or, $db\cos C+dc\cos B=0\quad\because\ d(\pi-A)=0.$

or, $\dfrac{db}{\cos B}+\dfrac{dc}{\cos C}=0.$

2·5. Partial derivatives of higher order.

The partial derivatives f_x and f_y are themselves functions of x and y and so may again admit of partial differentiation with respect to x and y.

The partial derivative of $f_x(x, y)$ with respect to x is denoted by f_{xx} or, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ or, $\frac{\partial^2 f}{\partial x^2}$.

Thus $$f_{xx}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}f_x(x, y)$$

$$=\underset{h\to 0}{Lt}\ \frac{f_x(x+h, y)-f_x(x, y)}{h}$$

Similarly, $$f_{yx}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial y}f_x(x, y)$$

$$=\underset{k\to 0}{Lt}\ \frac{f_x(x, y+k)-f_x(x, y)}{k}$$

$$f_{yy}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}f_y(x, y)$$

$$=\underset{k\to 0}{Lt}\ \frac{f_y(x, y+k)-f_y(x, y)}{k}$$

$$f_{xy}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}f_y(x, y)$$

$$=\underset{h\to 0}{Lt}\ \frac{f_y(x+h, y)-f_y(x, y)}{h}$$

The four partial derivatives are the second order derivatives of $f(x, y)$ and are also denoted by

$$\frac{\partial^2 f}{\partial x^2},\ \frac{\partial^2 f}{\partial y\,\partial x},\ \frac{\partial^2 f}{\partial y^2},\ \frac{\partial^2 f}{\partial x\,\partial y}.$$

The higher order partial derivatives are defined in the same way.

2.6. Commutative property of the order of partial derivatives for a function of two variables.

We have by definition

$$f_{xy}(a, b)=\left(\frac{\partial^2 f}{\partial x.\partial y}\right)_{(a, b)}$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)_{(a, b)},\ \text{at a point } (a, b)$$

$$=\frac{\partial}{\partial x}f_y(a, b)$$

$$=\underset{h\to 0}{Lt}\ \frac{f_y(a+h, b)-f_y(a, b)}{h} \qquad \cdots \qquad (1)$$

But $f_y(a+h, b)=\underset{k\to 0}{Lt}\ \frac{f(a+h, b+k)-f(a+h, b)}{k}$

And $f_y(a, b)=\underset{k\to 0}{Lt}\ \frac{f(a, b+k)-f(a, b)}{k}.$

Putting these values in (1) we get, $f_{xy}(a, b)$

$$=\underset{h\to 0}{Lt}\ \underset{k\to 0}{Lt}\left\{\frac{f(a+h, b+k)-f(a+h, b)-f(a, b+k)+f(a, b)}{hk}\right\}$$

$$=\underset{h\to 0}{Lt}\ \underset{k\to 0}{Lt}\ \frac{F(h, k)}{hk} \text{ (say.)} \qquad \cdots \qquad \cdots \qquad (2)$$

Similarly, it can be shown that

$$f_{yx}(a, b)=\underset{k\to 0}{Lt}\ \underset{h\to 0}{Lt}\ \frac{F(h, k)}{hk}. \qquad \cdots \qquad \cdots \qquad (3)$$

From (2) and (3) we find that $f_{xy}(a, b)$ and $f_{yx}(a, b)$ are repeated limits of the same function taken in different orders. But we know that the repeated limits may or may not be equal. So it follows that $f_{xy}(a, b)$ and $f_{yx}(a, b)$ may or may not be equal.

Illustrations :

Ex. 1. Let $f(x, y)=\frac{xy(x^2-y^2)}{x^2+y^2}$, $(x, y)\neq(0, 0)$ and $f(0, 0)=0$. Then show that $f_{xy}(0, 0)\neq f_{yx}(0, 0)$ *(C. H. 1964)*

$$f_{xy}(0, 0)=\underset{h\to 0}{Lt}\ \frac{f_y(0+h, 0)-f_y(0, 0)}{h} \qquad \cdots \qquad (1)$$

But $f_y(h, 0)=\underset{k\to 0}{Lt}\ \frac{f(h, 0+k)-f(h, 0)}{k}$

$$=\underset{k\to 0}{Lt}\ \frac{f(h, k)-f(h, 0)}{k}$$

$$= \underset{k\to 0}{Lt} \frac{1}{k}\left\{\frac{hk(h^2-k^2)}{h^2+k^2}-0\right\}$$

$$= \underset{k\to 0}{Lt} \frac{h(h^2-k^2)}{h^2+k^2} = \frac{h^3}{h^2} = h.$$

And $\quad f_y(0, 0) = \underset{k\to 0}{Lt} \dfrac{f(0, 0+k)-f(0, 0)}{k}$

$$= \underset{k\to 0}{Lt} \frac{1}{k}\{f(0, k)-f(0, 0)\}$$

$$= \underset{k\to 0}{Lt} \frac{1}{k}\{0-0\} = 0$$

$\therefore\quad$ From (1) we get

$$f_{xy}(0, 0) = \underset{h\to 0}{Lt} \frac{h-0}{h} = 1$$

Again $\quad f_{yx}(0, 0) = \underset{k\to 0}{Lt} \left\{\dfrac{f_x(0, 0+k)-f_x(0, 0)}{k}\right\} \quad \cdots \quad (2)$

But $\quad f_x(0, k) = \underset{h\to 0}{Lt} \dfrac{f(0+h, k)-f(0, k)}{h}$

$$= \underset{h\to 0}{Lt} \frac{f(h, k)-f(0, k)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{1}{h}\left\{\frac{hk(h^2-k^2)}{h^2+k^2}-0\right\}$$

$$= \underset{h\to 0}{Lt} \frac{k(h^2-k^2)}{h^2+k^2} = -\frac{k^3}{k^2} = -k$$

And $\quad f_x(0, 0) = \underset{h\to 0}{Lt} \dfrac{f(0+h, 0)-f(0, 0)}{h}$

$$= \underset{h\to 0}{Lt} \frac{1}{h}\{f(h, 0)-f(0, 0)\}$$

$$= \underset{h\to 0}{Lt} \frac{1}{h}\{0-0\} = 0.$$

$\therefore\quad$ From (2) we get

$$f_{yx}(0, 0) = \underset{k\to 0}{Lt} \left\{\frac{-k-0}{k}\right\} = -1$$

Hence, $\quad f_{xy}(0, 0) \neq f_{yx}(0, 0).$

Ex. 2. Let $\mathbf{f(x, y)=\dfrac{2xy}{\sqrt{x^2+y^2}},\ (x, y)\neq(0, 0)}$ **and** $\mathbf{f(0, 0)=0}$. **Then show that**

$$\mathbf{f_{xy}(0, 0)=f_{yx}(0, 0).}$$

$$f_{xy}(0, 0)=\underset{h\to 0}{Lt}\ \frac{f_y(0+h, 0)-f_y(0, 0)}{h} \quad \ldots \quad (1)$$

But $$f_y(h, 0)=\underset{k\to 0}{Lt}\ \frac{f(h, 0+k)-f(h, 0)}{k}$$

$$=\underset{k\to 0}{Lt}\ \frac{1}{k}\{f(h, k)-f(h, 0)\}$$

$$=\underset{k\to 0}{Lt}\ \frac{1}{k}\ \frac{2hk}{\sqrt{h^2+k^2}}=\underset{k\to 0}{Lt}\ \frac{2h}{\sqrt{h^2+k^2}}=2$$

And $$f_y(0, 0)=\underset{k\to 0}{Lt}\ \frac{f(0, 0+k)-f(0, 0)}{k}$$

$$=\underset{k\to 0}{Lt}\ \frac{1}{k}\{f(0, k)-f(0, 0)\}$$

$$=\underset{k\to 0}{Lt}\ \frac{1}{k}(0-0)=0$$

$\therefore$ From (1) we get

$$f_{xy}(0, 0)=\underset{h\to 0}{Lt}\ \frac{2-0}{h}=\infty$$

Again $$f_{yx}(0, 0)=\underset{k\to 0}{Lt}\ \frac{f_x(0, 0+k)-f_x(0, 0)}{k} \quad \ldots \quad (2)$$

But $$f_x(0, k)=\underset{h\to 0}{Lt}\ \frac{f(0+h, k)-f(0, k)}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{1}{h}\left[\frac{2hk}{\sqrt{h^2+k^2}}-0\right]$$

$$=\underset{h\to 0}{Lt}\ \frac{2k}{\sqrt{h^2+k^2}}=2$$

And $$f_x(0, 0)=\underset{h\to 0}{Lt}\ \frac{f(0+h, 0)-f(0, 0)}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{1}{h}(0-0)=0$$

$\therefore$ From (2) we get

$$f_{yx}(0, 0) = \underset{k \to 0}{Lt} \frac{2-0}{k} = \infty .$$

Hence, $f_{xy}(0, 0) = f_{yx}(0, 0)$.

Ex. 3. **Let** $f(x, y) = \begin{cases} xy \text{ if } |x| \geqslant |y| \\ -xy \text{ if } |x| < |y| \end{cases}$

Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ *(C. H. (old) 1963)*

$$f_{xy}(0, 0) = \underset{h \to 0}{Lt} \frac{f_y(0+h, 0) - f_y(0, 0)}{h} \quad \cdots \quad (1)$$

But $f_y(h, 0) = \underset{k \to 0}{Lt} \frac{f(h, 0+k) - f(h, 0)}{k}$

$$= \underset{k \to 0}{Lt} \frac{1}{k} \{hk - 0\} \qquad \because \begin{array}{l} h > k \\ |x| > |y| \end{array}$$

$$= h$$

And $f_y(0, 0) = \underset{k \to 0}{Lt} \frac{f(0, 0+k) - f(0, 0)}{k}$

$$= \underset{k \to 0}{Lt} \frac{1}{k} \{0 - 0\} = 0$$

$\therefore$ From (1) $f_{xy}(0, 0) = \underset{h \to 0}{Lt} \frac{h-0}{h} = 1.$

Again $f_{yx}(0, 0) = \underset{k \to 0}{Lt} \frac{f_x(0, 0+k) - f_x(0, 0)}{k} \quad \cdots \quad (2)$

But $f_x(0, k) = \underset{h \to 0}{Lt} \frac{f(0+h, k) - f(0, k)}{h} \qquad \because \begin{array}{l} k > h \\ |y| > |x| \end{array}$

$$= \underset{h \to 0}{Lt} \frac{1}{h} \{-hk - 0\} = -k$$

And $f_x(0, 0) = \underset{h \to 0}{Lt} \frac{f(0+h, 0) - f(0, 0)}{h}$

$$= \underset{h \to 0}{Lt} \frac{1}{h} \{0 - 0\} = 0$$

$\therefore$ From (2)

$$f_{yx} = \underset{k \to 0}{Lt} \frac{-k-0}{k} = -1$$

Hence, $f_{xy}(0, 0) \neq f_{yx}(0, 0)$

2.7. Reversal order of derivation : Sufficient conditions.

1. Schwarz's Theorem. Form—I. Let $f(x, y)$ be defined in some neighbourhood of (x, y). If its derivatives (*i*) f_x, f_y, f_{xy} and f_{yx} all exist in that neighbourhood of (x, y) and

(*ii*) f_{xy} (or f_{yx}) is continuous at the point (x, y)

then $\dfrac{\partial^2 y}{\partial x \partial y} = \dfrac{\partial^2 y}{\partial y \partial x}$.

Proof. Let us consider the points $(x+h, y)$, $(x, y+k)$ and $(x+h, y+k)$ in the neighbourhood of (x, y) so that h, k are small enough $\neq 0$. Then in this neighbourhood of (x, y) $\{f(x+h, y+k) - f(x, y+k)\} - \{f(x+h, y) - f(x, y)\}$ will be a function of (h, k) only. So we can write

$$F(h, k) = \{f(x+h, y+k) - f(x, y+k)\} - \{f(x+h, y) - f(x, y)\}$$

$$= \phi(y+k) - \phi(y) \qquad \cdots \qquad \cdots \qquad (1)$$

if we write, $\phi(y) = f(x+h, y) - f(x, y)$

Since f_y exists in some neighbourhood of (x, y), $\phi(y)$ is derivable in that neighbourhood of (x, y)

$$\therefore \quad \phi'(y) = f_y(x+h, y) - f_y(x, y)$$

So at the point $y = y + \theta k$

$$\phi'(y+\theta k) = f_y(x+h, y+\theta k) - f_y(x, y+\theta k) \qquad \cdots \qquad (2)$$

Now applying Mean Value Theorem in (1)

$$F(h, k) = k\phi'(y+\theta k) \text{ where } 0 < \theta < 1$$

$$= k\{f_y(x+h, y+\theta k) - f_y(x, y+\theta k)\} \text{ from (2)}, \cdots \quad (3)$$

Since f_{xy} is continuous at the point (x, y), f_{xy} exists in some neighbourhood of (x, y)

$\therefore$ $f_y(x, y+\theta k)$ is derivable with respect to x in $(x, x+h)$

$\therefore$ Applying Mean Value Theorem

$$f_y(x+h,\ y+\theta k)-f_y(x,\ y+\theta k)$$
$$=h\ f_{xy}(x+\theta'h,\ y+\theta k) \quad \text{where} \quad 0<\theta<1$$
$$0<\theta'<1$$

$\therefore$ From (3)

$$F(h,\ k)=khf_{xy}(x+\theta'h,\ y+\theta k)$$

or, $$f_{xy}(x+\theta'h,\ y+\theta k)=\frac{F(h,\ k)}{hk} \tag{4}$$

Since f_{xy} is continuous at $(x,\ y)$

$$\underset{h\to 0}{Lt}\ \underset{k\to 0}{Lt}\ f_{xy}(x+\theta'h,\ y+\theta k)=f_{xy}(x,\ y).$$

Also $\underset{h\to 0}{Lt}\ \underset{k\to 0}{Lt}\ \dfrac{F(h,\ k)}{hk}$

$$=\underset{h\to 0}{Lt}\ \underset{k\to 0}{Lt}\ \frac{1}{k}\left\{\frac{f(x+h,\ y+k)-f(x,\ y+k)}{h}\right.$$
$$\left.-\frac{f(x+h,\ y)-f(x,\ y)}{h}\right\}$$

$$=\underset{k\to 0}{Lt}\ \frac{1}{k}\left\{\underset{h\to 0}{Lt}\ \frac{f(x+h,\ y+k)-f(x,\ y+k)}{h}\right.$$
$$\left.-\underset{h\to 0}{Lt}\ \frac{f(x+h,\ y)-f(x,\ y)}{h}\right\}$$

$$=\underset{k\to 0}{Lt}\ \frac{f_x(x,\ y+k)-f_x(x,\ y)}{k}$$

$$=f_{yx}(x,\ y).$$

Hence from (4)

$$f_{xy}=f_{yx}$$

Schwarz's Theorem. Form II. If $f_y(x, y)$ and $f_{yx}(x, y)$ exist in a certain neighbourhood of (a, b) and if $f_{yx}(x, y)$ is c ntinuous at (a, b) then $f_{xy}=f_{yx}$ at (a, b).

Proof. Since f_{yx} is continuous at (a, b), f_x exists in the neighbourhood of (a, b). So the given conditions imply that $f_x(x, y)$, $f_y(x, y)$ and $f_{yx}(x, y)$ exist at every point (x, y)

of a certain neighbourhood of (a, b). Let $(a+h, b)$, $(a+h, b+k)$ and $(a, b+k)$ be three neighbouring points of (a, b) where h, k are very small $\neq 0$. Then we can write

$$F(h, k)=f(a+h, b+k)-f(a+h, b)-f(a, b+k)+f(a, b) \cdots (1)$$

Now denoting $\phi(x)=f(x, b+k)-f(x, b)$ $\cdots$ $\cdots(2)$

we get

$$F(h, k)=\phi(a+h)-\phi(a) \quad \cdots \quad \cdots \quad \cdots(3)$$

Since f_x exists in the neighbourhood of (a, b), the function $\phi(x)$ is derivable with respect to x in the closed interval $(a, a+h)$.

$\therefore$ applying the Mean Value Theorem to the R. H. S. of (3)

$$F(h, k)=h\phi'(a+\theta h) \text{ where } 0<\theta<1$$
$$=h[f_x(a+\theta h, b+k)-f_x(a+\theta h, b] \quad \cdots \quad (4)$$
[using (2)]

Since f_{yx} exists in the neighbourhood of (a, b), the function $f_x(a+\theta h, y)$ is derivable with respect to y in the closed interval $(b, b+k)$

$\therefore$ Applying the mean value theorem to the R. H. S. of (4)

$$F(h, k)=hk\, f_{yx}(a+\theta h, b+\theta' k) \text{ where } 0<\theta<1 \text{ and } 0<\theta'<1$$

or, $$\frac{F(h, k)}{hk}=f_{yx}(a+\theta h, b+\theta' k) \quad \cdots \quad \cdots \quad (5)$$

Now $$\frac{F(h, k)}{hk}=\frac{1}{h}\left[\frac{f(a+h, b+k)-f(a+h, b)}{k} - \frac{f(a, b+k)-f(a, b)}{k}\right]$$

$$\therefore \underset{h\to 0}{Lt}\ \underset{k\to 0}{Lt}\ \frac{F(h, k)}{hk}$$
$$=\underset{h\to 0}{Lt}\ \frac{1}{h}\left[\underset{k\to 0}{Lt}\ \frac{f(a+h, b+k)-f(a+h, b)}{k} - \frac{f(a, b+k)-f(a, b)}{k}\right]$$

$$=\underset{h\to 0}{Lt}\ \frac{f_y(a+h,\ b)-f_y(a,\ b)}{h}$$

$$=f_{xy}(a,\ b)$$

Also $\underset{h\to 0}{Lt}\ \underset{k\to 0}{Lt}\ f_{yx}(a+\theta h,\ b+\theta' k)=f_{yx}(a,\ b)$

$\because$ f_{yx} is continuous at $(a,\ b)$

Hence from (5) we get

$$f_{xy}(a,\ b)=f_{yx}(a,\ b).$$

Schwarz's Theorem. Form III.

If $f_{xy}(x,\ y)$ and $f_{yx}(x,\ y)$ are both continuous at (a, b) then

$$f_{xy}(a,\ b)=f_{yx}(a,\ b).$$

Since f_{xy} and f_{yx} are both continuous at $(a,\ b)$, f_y and f_x exist in some neighbourhood of $(a,\ b)$

So the given conditions imply that f_x, f_y and f_{xy} exists at every point $(x,\ y)$ of a certain neighbourhood cf $(a,\ b)$ and it is the same as form II above.

Young's Theorem.

If f_x and f_y exists in a certain neighbourhood of (a, b) and if f_x and f_y are both differentiable at (a, b), then

$$f_{xy}=f_{yx} \text{ at } (a,\ b).$$

Proof. The given conditions imply that $f^2{}_x$, f_{yx}, $f^2{}_y$, f_{xy} exist at $(a,\ b)$. Let $(a+h,\ b)$, $(a+h,\ b+h)$ and $(a,\ b+h)$ be three points in the neighbourhood of $(a,\ b)$ so that h is very small $\neq 0$. Then we can write

$$F(h,\ h)=f(a+h,\ b+h)-f(a+h,\ h)-f(a,\ b+h)+f(a,\ b) \qquad \cdots \quad (1)$$

If then $\phi(x)=f(x,\ b+h)-f(x,\ b)$ $\qquad \cdots \quad \cdots \quad (2)$

we get

$$F(h,\ h)=\phi(a+h)-\phi(a) \qquad \cdots \quad \cdots \quad (3)$$

Since f_x exists in the neighbourhood of $(a,\ b)$, $\phi(x)$ is derivable with respect to x in the closed interval $(a,\ a+h)$.

So applying the Mean Value Theorem to the R. H. S. of (3)

$$F(h, h)=h\phi'(a+\theta h),\ 0<\theta<1$$
$$=h[f_x(a+\theta h, b+h)-f_x(a+\theta h, b)] \quad \cdots \quad (4)$$
[using (2)]

Now since $f_x(a, b)$ is differentiable at (a, b), we have using the condition of differentiability

$$f_x(a+\theta h, b+h)-f_x(a, b)=\theta h\, f_x{}^2(a, b)+hf_{yx}(a, b)$$
$$+\theta h\, \phi_1(h, h)+h\psi_1(h, h) \quad \cdots \quad (5)$$

and $f_x(a+\theta h, b)-f_x(a, b)=\theta hf^2{}_x(a, b)+\theta h\, \phi_2(h, h) \quad \cdots \quad (6)$

where ϕ_1, ϕ_2 and ψ_1 all $\to 0$ as $h\to 0$.

Putting (5) and (6) in (4)

$$\frac{F(h, h)}{h^2}=f_{yx}(a, b)+\theta\phi_1(h, h)+\psi_1(h, h)-\theta\phi_2(h, h) \quad \cdots \quad (7)$$

Similarly, writing

$\psi(y)=f(a+h, y)-f(a, y)$ and proceding as before we shall get

$$\frac{F(h, h)}{h^2}=f_{xy}(a, b)+\theta'\psi_2(h, h)+\phi_3(h, h)-\theta'\psi_3(h, h) \quad \cdots \quad (8)$$

Since ϕ_1, ϕ_2, ϕ_3, ψ_1, ψ_2 all tends to zero as h tends to zero we get in the limits as $h\to 0$ from (7) and (8)

$$f_{yx}(a, b)=f_{xy}(a, b).$$

Exercise 2

1. If da, db, dc be the errors in measuring the sides a, b, c of the triangle ABC, then the error $d\triangle$ of the area $\triangle$ of the triangle is given by

$$d\triangle=\frac{abc}{4\triangle}(\cos A\; da+\cos B\; db+\cos C\; dc).$$

2. If da, db, dc be the errors in measuring the sides a, b, c of a triangle ABC, show that

$$dA=\tfrac{1}{2}\, a\, (da-db\cos C-dc\cos B)/\triangle$$

where $\triangle$ is the area of the triangle and verify that

$$dA+dB+dC=0.$$

3. Show that (*a*) $\sqrt{|xy|}$ is not differentiable at the origin.

(*b*) $|x|+|y|$ is continuous but not differentiable at the origin.

4. For the function

$$f(x, y)=xy\frac{x^2+a^2y^2}{x^2+y^2}, \ (x^2+y^2\neq 0 \text{ and } a\neq 0)$$

and $f(0, 0)=0$

show that $f_{xy}(0, 0)\neq f_{yx}(0,0)$ (C. H 1960)

5. If $f(x, y)=x^2 \tan^{-1}\frac{y}{x}-y^2 \tan^{-1}\frac{x}{y}$, $xy\neq 0$

and $=0$ elsewhere

show that $f_{xy}(0, 0)\neq f_{yx}(0, 0)$

6. If $f(x, y)=\dfrac{x^3+y^3}{x-y}$, $x\neq y$

and $=0$ when $x=y$

show that $f_x(0, 0)$ and f_y $(0, 0)$ both exist but $f(x, y)$ is discontinuous at the origin.

7. Show that the function defined by

$$f(x, y)=x^2 \tan^{-1}\frac{y}{x}-y^2 \tan^{-1}\frac{x}{y}$$

and $f(0, y)=f(x, 0)=0$

satisfies $f_{xy}=f_{yx}$ for all point except at $(0, 0)$ (C. H. 1964)

8. If the mixed partial derivatives f_{xy} and f_{yx} of a function $f(x, y)$ are continuous in a region R, then show that

$$f_{xy}=f_{yx}$$

holds throughout the interior of the region R.

9. For the function

$$f(x, y)=\frac{1}{\sqrt{y}}e^{-(x-a)^2/4y}$$

show that $f_{xy}=f_{yx}$

and $f_{xx}=f_y$,

10. For the function

$$f(x, y)=\sqrt{x^2+y^2}\sin 2\phi \text{ where } f(0, 0)=0$$

and $\phi=\tan^{-1}\left(\dfrac{y}{x}\right)$. Show that

$$f_{xy}(0, 0)=f_{yx}(0, 0)$$

CHAPTER III

HOMOGENEOUS FUNCTIONS

3.1. A function $f(x, y)$ is said to be homogeneous of degree n if it can be expressed in the form $x^n\phi\left(\frac{y}{x}\right)$ or in the form $y^n\psi\left(\frac{x}{y}\right)$.

For example consider $f(x, y)=\dfrac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}$

Hence $$f(x, y)=\frac{x^{\frac{1}{4}}\left\{1+\left(\frac{y}{x}\right)^{\frac{1}{4}}\right\}}{x^{\frac{1}{5}}\left\{1+\left(\frac{y}{x}\right)^{\frac{1}{5}}\right\}}=x^{\frac{1}{20}}\left\{1+\left(\frac{y}{x}\right)^{\frac{1}{4}}\right\}\left\{1+\left(\frac{y}{x}\right)^{\frac{1}{5}}\right\}^{-1}$$

$$=x^{\frac{1}{20}}\phi\left(\frac{y}{x}\right)$$

$\therefore$ $f(x, y)$ is a homogeneous function of degree 1/20.

Again a function $f(x, y)$ can always be found to satisfy an identity of the form

$$f(tx, ty)=t^n f(x, y)$$

for all positive values of t. So alternatively, we can say that, a function $f(x, y)$ is said to be homogeneous of degree n if it can be expressed in the form

$$f(tx, ty)=t^n f(x, y) \text{ for } all \text{ positive}$$

values of t.

For examples

$f(x, y)=\dfrac{x+y}{2x+3y}$ is a homogeneous function of degree zero. For

$$f(tx, ty)=\frac{tx+ty}{2tx+3ty}=t^0\,\frac{x+y}{2x+3y}=t^0 f(x, y).$$

A function $f(x, y, z)$ is said to be homogeneous of degree n if it can be expressed in the form

$$x^n\phi\left(\frac{y}{x}, \frac{z}{x}\right) \quad \text{or,} \quad y^n\psi\left(\frac{x}{y}, \frac{z}{y}\right) \quad \text{or,} \quad z^n\zeta\left(\frac{x}{z}, \frac{y}{z}\right).$$

Also alternatively, a function $f(x, y, z)$ is said to be homogeneous of degree n if it can be expressed in the form

$$f(tx, ty, tz)=t^n f(x, y, z)$$

for all positive values of t.

Note. If u be a homogeneous function of degree n in x, y, z, then each of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ is a homogeneous function of degree $(n-1)$.

3.2. Euler's Theorem on Homogeneous functions of two variables.

If $f(x, y)$ be a homogeneous function of x and y of degree n, then

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=n.f(x, y).$$

Proof. Since $f(x, y)$ is a homogeneous function of degree n, we can write

$$f(x, y)=x^n\phi\left(\frac{y}{x}\right)$$

$$=x^n\phi(v) \text{ if we put } v=\frac{y}{x}.$$

$$\therefore \quad \frac{\partial f}{\partial x}=nx^{n-1}\phi(v)+x^n\phi'(v)\frac{\partial v}{\partial x}$$

$$=nx^{n-1}\phi(v)+x^n\phi'(v)\left(-\frac{y}{x^2}\right)$$

$$\left[\because \quad v=\frac{y}{x} \quad \frac{\partial v}{\partial x}=-\frac{y}{x^2}.\right]$$

$$\therefore \quad x\frac{\partial f}{\partial x}=nx^n\phi(v)-x^{n-1}y\phi'(v) \qquad \dots \qquad \dots \qquad (1)$$

Also $\dfrac{\partial f}{\partial y}=x^n\phi'(v)\dfrac{\partial v}{\partial y}$

$$=x^n\phi'(v)\cdot\frac{1}{x}$$

$$\left[\because\quad v=\frac{y}{x}\quad \frac{\partial v}{\partial y}=\frac{1}{x}\right]$$

$$\therefore\quad y\frac{\partial f}{\partial y}=x^{n-1}y\phi'(v) \qquad \cdots \qquad \cdots \qquad (2)$$

Adding (1) and (2)

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nx^n\phi(v)$$

$$=nx^n\phi\left(\frac{y}{x}\right)=n.f(x,\ y).$$

Cor. If u be a homogeneous function of x and y of degree n. then

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2u=n(n-1)u$$

where $\left(x\dfrac{\partial}{\partial x}+y\dfrac{\partial}{\partial y}\right)^2u=x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\,\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}.$

Since u is a homogeneous function of degree n, each of $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$ is a homogeneous function of degree $(n-1)$.

So by Euler's theorem

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\frac{\partial u}{\partial x}=(n-1)\frac{\partial u}{\partial x} \qquad \cdots \qquad \cdots \qquad (1)$$

and $$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\frac{\partial u}{\partial y}=(n-1)\frac{\partial u}{\partial y}. \qquad \cdots \qquad \cdots \qquad (2)$$

Multiplying (1) by x and (2) by y and adding

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x.\partial y}$$

$$=(n-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=(n-1)nu$$

$$\therefore\quad \left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial u}\right)^2u=n(n-1)u.$$

3.3. Composite functions.

Let $\quad z=f(x, y)$

where $x=\phi(t), \quad y=\psi(t)$.

Then z is said to be a composite function of t.

Again, if $z=f(x, y)$

where $\quad x=\phi(u, v),\ y=\psi(u, v)$

Then z is said to be composite function of u and v.

3.4. Differentiation of composite functions.

Let $z=f(x, y)$

possess continuous partial derivatives and

let $\quad x=\phi(t)$

$\quad\quad y=\psi(t)$

possess continuous derivatives.

Then $\quad \dfrac{dz}{dt}=\dfrac{\partial z}{\partial x}\cdot\dfrac{dx}{dt}+\dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$

Proof. The given conditions imply that f_x, f_y and $\phi'(t)$, $\psi'(t)$ are continuous.

When t changes to $t+\delta t$, let x and y change to $x+\delta x$ and $y+\delta y$ respectively.

$$\therefore\quad x+\delta x=\phi(t+\delta t),\quad y+\delta y=\psi(t+\delta t)$$

$$\therefore\quad z+\delta z=f(x+\delta x,\ y+\delta y)$$

or $\quad \delta z=f(x+\delta x, y+\delta y)-f(x, y) \qquad \because\ z=f(x, y)$

$$=[f(x+\delta x, y+\delta y)-f(x, y+\delta y)]$$
$$+[f(x, y+\delta y)-f(x, y)]$$

Applying M. V. T.

$$=\delta x.f_x(x+\theta_1\delta x,\ y+\delta y)+\delta y f_y(x,\ y+\theta_2\delta y)$$

for $(0<\theta_1, \theta_2<1)$

or $\quad \dfrac{\delta z}{\delta t}=\dfrac{\delta x}{\delta t}f_x(x+\theta_1\delta x,\ y+\delta y)+\dfrac{\delta y}{\delta t}f_y(x,\ y+\theta_2\delta y) \quad \cdots \quad (1)$

Let $\delta t \to 0$, so that δx and $\delta y \to 0$

Then $\qquad \underset{\delta t \to 0}{Lt} f_x(x+\theta_1 \delta x,\ y+\delta y)$

$$= \underset{(\delta x,\ \delta y)\to(0,\ 0)}{Lt} f_x(x+\theta_1\delta x,\ y+\delta y)$$

$$= f_x(x,\ y) \qquad [\because\ f_x \text{ is continuous}]$$

$$= \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}.$$

and $\qquad \underset{\delta t \to 0}{Lt} f_y(x,\ y+\theta_2\delta y) = \underset{\delta y \to 0}{Lt} f_y(x,\ y+\theta_2\delta y)$

$$= f_y(x,\ y) \qquad [\because\ f_y \text{ is continuous}]$$

$$= \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}.$$

Hence, from (1) proceeding to the limit as $\partial t \to 0$ we get

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}.$$

Cor 1. The last result may be written as

$$\frac{dz}{dt}dt = \frac{\partial z}{\partial x}\frac{dx}{dt}dt + \frac{\partial z}{\partial y}\frac{dy}{dt}dt. \qquad \cdots \qquad (1)$$

But the differential dx, dy and dz are given by

$$dx = \frac{dx}{dt}dt, \quad dy = \frac{dy}{dt}dt, \quad dz = \frac{dz}{dt}at.$$

Hence from (1) we get

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Cor 2. Generally, for functions of several variables $u = f(x,\ y,\ z\cdots)$ where each of x, y, z $\cdots$ is a function of t

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} + \cdots$$

$\because$ and $\qquad du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \cdots$

Cor 3. $z=f(x, y)$, let $y=\phi(x)$,

then z is a function of x alone.

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}\frac{dx}{dx}+\frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}. \tag{1}$$

If in particular $z=f(x, y)=0$

Then $\frac{dz}{dx}=0$

$\therefore$ From (1), we get

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}=0$$

or, $$\frac{dy}{dx}=-\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}$$

This gives $\frac{dy}{dx}$ for implicit function $f(x, y)=0$.

3.5. Implicit Functions.

If from an equation $f(x, y)=0$, it is not possible to solve for y in terms of x, then $f(x, y)$ is called an implicit function.

If on the other band, it is possible to solve for y in terms of x i.e., it is possible to find a relation of the form $y=\phi(x)$, then $f(x, y)$ is called an explicit function.

As for example $x^2+y^2+2=0$ is an implicit equation for from it $y=\pm\sqrt{-2-x^2}$ and so y cannot be obtained for any value of x.

The equation $x^2+y^2=0$ gives $y=\pm\sqrt{-x^2}$ which gives $y=0$ for $x=0$. But for $x\neq0$, y can not be obtained. So $x^2+y^2=0$ is an implicit equation for $x\neq0$ and an explicit equation for $x=0$.

Condition for the existence of an explicit function $y=\phi(x)$ from an implicit relation $f(x, y)=0$.

Theorem :

If $f(x, y)$ be continuous together with its first partial derivatives in the neighbourhood of a point (x_0, y_0) where $f(x_0, y_0)=0$ and $f_y(x_0, y_0)\neq 0$, then there exists a function $y=\phi(x)$ continuous in the neighbourhood of x_0 satisfying $y_0=\phi(x_0)$ such that $f\{x, \phi(x)\}=0$.

3.6. Differentiation of Implicit Functions.

Let $f(x, y)=0$ be an equation of an implicit function where y is a differentiable function of x and the partial derivatives f_x and f_y are continuous.

Then $\dfrac{dy}{dx}=-f_x/f_y$.

$$\because \quad f(x, y)=0$$

$$f(x+\delta x, y+\delta y)=0$$

$$\therefore \quad f(x+\delta x, y+\delta y)-f(x, y)=0.$$

$$\text{or,} \quad f(x+\delta x, y+\delta y)-f(x, y+\delta y)+f(x, y+\delta y) -f(x, y)=0 \quad \cdots \quad (1)$$

Now by the Mean Value Theorem

$$f(x+\delta x, y+\delta y)-f(x, y+\delta y)=\delta x\frac{\delta}{\delta x}f(x+\theta_1\delta x, y+\delta y) \quad \text{for } 0<\theta_1<1$$

$$\text{and} \quad f(x, y+\delta y)-f(x, y)=\delta y\frac{\delta}{\delta y}f(x, y+\theta_2\delta y) \quad \text{for } 0<\theta_2<1$$

So from (1)

$$\delta x.\frac{\partial}{\partial x}f(x+\theta_1\delta x, y+\delta y)+\delta y\frac{\delta}{\delta y}f(x, y+\theta_2\delta y)=0$$

$$\text{or,} \quad \frac{\partial}{\partial x}f(x+\theta_1\delta x, y+\delta y)+\frac{\delta y}{\delta x}\frac{\delta}{\delta y}f(x, y+\theta_2\delta y)=0 \quad \cdots \quad (2)$$

Since y is a differentiable function of x, when $\delta x\to 0$,

δy also $\to 0$. Also since f_x and f_y are continuous, we get from (2) after proceeding to the limit as $\delta x \to 0$

$$\frac{\partial f}{\partial x}+\frac{dy}{dx}\cdot\frac{\partial f}{\partial y}=0$$

or $$\frac{dy}{dx}=-\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}},\quad \frac{\partial f}{\partial y}\neq 0$$

3.7. Partial Derivatives of a function of two functions.

Let $z=f(x, y)$

possess continuous first order partial derivatives w. r. to x, y.

Let $x=\phi(u, v)$

$y=\psi(u, v)$

possess continuous first order partial derivatives. Then

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}.$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}.$$

Proof. Since $z=f(x, y)$ possess continuous first order partial derivatives

$$dz=\frac{\partial z}{\partial x}\,dx+\frac{\partial z}{\partial y}\,dy \qquad \cdots \qquad (1)$$

Since $x=\phi(u, v)$ and $y=\psi(u, v)$ possess continuous first order partial derivations

$$dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv$$

and $$dy=\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv$$

Putting these values of dx and dy in (1) we get

$$dz=\frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right)+\frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv\right)$$

$$=\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\right)du+\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\right)dv. \qquad \cdots \quad (2)$$

Again $\because \quad z=f(x, y)$

$$=f\{\phi(u, v), \psi((u, v)\}$$

$$=F(u, v)$$

$$dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv \qquad \cdots \qquad \cdots \qquad (3)$$

Comparing (2) and (3), since du and dv are independent, we get

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

and $$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Note : Similarly, for more than two variables

i.e., if $z=f(x_1, x_2, x_3 \cdots)$ and $x_1=f_1(u_1, u_2, u_3 \cdots)$

$x_2=f_2(u_1, u_2, u_3 \cdots)$

$x_3=f_3(u_1, u_2, u_3 \cdots)$ and so on

Then $$\frac{\partial z}{\partial u_1}=\frac{\partial z}{\partial x_1}\cdot\frac{\partial x_1}{\partial u_1}+\frac{\partial z}{\partial x_2}\cdot\frac{\partial x_2}{\partial u_1}+\frac{\partial z}{\partial x_3}\cdot\frac{\partial x_3}{\partial u_1}+\cdots$$

$$\frac{\partial z}{\partial u_2}=\frac{\partial z}{\partial x_1}\cdot\frac{\partial x_1}{\partial u_2}+\frac{\partial z}{\partial x_2}\cdot\frac{\partial x_2}{\partial u_2}+\frac{\partial z}{\partial x_3}\cdot\frac{\partial x_3}{\partial u_2}+\cdots$$

$$\frac{\partial z}{\partial u_3}=\frac{\partial z}{\partial x_1}\cdot\frac{\partial x_1}{\partial u_3}+\frac{\partial z}{\partial x_2}\cdot\frac{\partial x_2}{\partial u_3}+\frac{\partial z}{\partial x_3}\cdot\frac{\partial x_3}{\partial u_3}+\cdots$$

and so on.

Cor. 1. If $f=f(x, y, z)$, $g=g(x, y, z)$ are differentiable functions of x, y, z and $x=x(p, q)$, $y=y(p, q)$, $z=z(p, q)$ are differentiable functions of p, q, then (C.H. 1969)

$$\begin{pmatrix} f_p & f_q \\ g_p & g_q \end{pmatrix}=\begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \end{pmatrix}\begin{pmatrix} x_p & x_q \\ y_p & y_q \\ z_p & z_q \end{pmatrix}$$

$$\text{L.H.S} = \begin{pmatrix} f_x x_p + f_y y_p + f_z z_p & f_x x_q + f_y y_q + f_z z_q \\ g_x x_p + g_y y_p + g_z z_p & g_x x_q + g_y y_q + g_z z_q \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial p} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial p} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial p} & \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial q} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial q} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial q} \\ \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial p} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial p} + \frac{\partial g}{\partial z} \cdot \frac{\partial z}{\partial p} & \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial q} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial q} + \frac{\partial g}{\partial z} \cdot \frac{\partial z}{\partial q} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial p} & \frac{\partial g}{\partial q} \end{pmatrix} = \begin{pmatrix} f_p & f_q \\ g_p & g_q \end{pmatrix} \text{ from the given conditions.}$$

Cor. 2. If $y_1 = f_1(x_1, x_2, x_3)$ and $y_2 = f_2(x_1, x_2, x_3)$ are differentiable functions of x_1, x_2, x_3 in a region R and $x_1 = \phi_1(t_1, t_2)$, $x_2 = \phi_2(t_1, t_2)$, $x_3 = \phi_3(t_1, t_2)$ are differentiable functions of t_1, t_2 in a region S, then (C. H. 1967)

$$\begin{pmatrix} \frac{\partial y_1}{\partial t_1} & \frac{\partial y_1}{\partial t_2} \\ \frac{\partial y_2}{\partial t_1} & \frac{\partial y_2}{\partial t_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} \\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} \\ \frac{\partial x_3}{\partial t_1} & \frac{\partial x_3}{\partial t_2} \end{pmatrix}$$

$$\text{R.H.S.} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_1} + \frac{\partial y_1}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_1} + \frac{\partial y_1}{\partial x_3} \cdot \frac{\partial x_3}{\partial t_1}, & \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_2} + \frac{\partial y_1}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_2} + \frac{\partial y_1}{\partial x_3} \cdot \frac{\partial x_3}{\partial t_2} \\ \frac{\partial y_2}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_1} + \frac{\partial y_2}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_1} + \frac{\partial y_2}{\partial x_3} \cdot \frac{\partial x_3}{\partial t_1}, & \frac{\partial y_2}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_2} + \frac{\partial y_2}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_2} + \frac{\partial y_2}{\partial x_3} \cdot \frac{\partial x_3}{\partial t_2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial y_1}{\partial t_1} & \frac{\partial y_1}{\partial t_2} \\ \frac{\partial y_2}{\partial t_1} & \frac{\partial y_2}{\partial t_2} \end{pmatrix} \text{ from the given conditions.}$$

3.8. Space derivatives. Differentiation in a given direction.

So far we have seen that a differentiable function $f(x, y)$ have partial derivatives in the directions parallel to

the x and y axis. Now we shall show that the differentiable function can have derivative in any direction at an angle α to the positive direction of the x axis.

Suppose, the point $(x+h, y+k)$ approaches the point (x, y) along a line through (x, y) making a constant angle α with the positive direction of x axis. Then h and k will not tend to zero independently of another and will satisfy simultaneously the relation

$$h = r\cos\alpha, \quad k = r\sin\alpha$$

so that $r = \sqrt{h^2+k^2}$.

Then $\underset{r\to 0}{Lt} \dfrac{f(x+r\cos\alpha, y+k\sin\alpha) - f(x, y)}{r}$

if exists is called the derivative of $f(x, y)$ in the direction α and is denoted by $D_{(\alpha)}f(x, y)$

Thus the derivative in the direction α is

$$D_{(\alpha)}f(x, y) = \underset{r\to 0}{Lt} \frac{f(x+r\cos\alpha, y+r\sin\alpha) - f(x, y)}{r}.$$

From this, we can easily deduce the derivative along x axis i.e., for $\alpha = 0$, also the derivative along y axis

i.e., for $\alpha = \dfrac{\pi}{2}$.

$$\text{Thus} \quad D_{(0)}f(x, y) = \underset{r\to 0}{Lt} \frac{f(x+r, y) - f(x, y)}{r}$$

$$= \frac{\partial f}{\partial x}.$$

$$\text{and} \quad D_{\left(\frac{\pi}{2}\right)} f(x, y) = \underset{r\to 0}{Lt} \frac{f(x, y+r) - f(x, y)}{r}$$

$$= \frac{\partial f}{\partial y}.$$

Note : Since partial derivatives have both magnitude and direction, it is a vector quantity. Thus if $\vec{G}$ be a vector whose components are f_x and f_y along two rectangular axes x and y respectively, then

$\vec{G} = if_x + jf_y$ where i, j are the unit vectors along the axes respectively.

Theorem :

Let f(x, y) be a differentiable function of x and y, $\vec{a}$ a vector whose components are cos α, sin α, and $\vec{G}$, a vector whose components are f_x, f_y along two perpendicular axes. Then

$$\vec{G}\cdot\vec{a}=\underset{r\to 0}{\mathrm{Lt}}\ \frac{f(x+r\cos\alpha,\ y+r\sin\alpha)-f(x,y)}{r}$$

we have $\vec{G}=if_x+jf_y$

and $\vec{a}=i\cos\alpha+j\sin\alpha$

$$\therefore\quad \vec{G}.\vec{a}=(if_x+jf_y).(i\cos\alpha+j\sin\alpha)$$
$$=f_x\cos\alpha+f_y\sin\alpha \quad \cdots \quad \cdots \quad (1)$$

Again since $f(x, y)$ is a differentiable function of (x, y) we must have

$$f(x+r\cos\alpha,\ y+r\sin\alpha)-f(x,y)=r\cos\alpha\,.\,f_x(x,y)$$
$$+r\sin\alpha\,.\,f_y(x,y)+r\cos\alpha\,.\,\phi(r\cos\alpha,\ r\sin\alpha)$$
$$+r\sin\alpha\,.\,\psi(r\cos\alpha,\ r\sin\alpha)$$

where $\phi(r\cos\alpha, r\sin\alpha)=0$ and $\psi(r\cos\alpha, r\sin\alpha)=0$.

or, $$\frac{f(x+r\cos\alpha,\ y+r\sin\alpha)-f(x,y)}{r}=\cos\alpha\,.\,f_x(x,y)$$
$$+\sin\alpha\,.\,f_y(x,y)+\cos\alpha\,.\,\phi(r\cos\alpha,\ r\sin\alpha)$$
$$+\sin\alpha\,.\,\psi(r\cos\alpha,\ r\sin\alpha).$$

Now proceeding to the limit as $r\to 0$, we get

$$\underset{r\to 0}{Lt}\ \frac{f(x+r\cos\alpha,\ y+r\sin\alpha)-f(x,y)}{r}$$
$$=\cos\alpha\,.\,f_x(x,y)+\sin\alpha\,.\,f_y(x,y)$$
$$=f_x\cos\alpha+f_y\sin\alpha$$
$$=\vec{G}.\vec{a}\quad\text{by (1).}$$

3.9. Euler's Theorem for a function of three variables.

If $f(x, y, z)$ be a homogeneous function in x, y, z of degree n, possessing continuous partial derivatives, then

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=n\,.\,f(x,y,z).$$

Proof. Since $f(x, y, z)$ is a homogeneous function in x, y, z of degree n, we can write

$$f(tx, ty, tz)=t^n . f(x, y, z) \cdots \quad \cdots \quad (1)$$

for all values of t.

Now replacing tx, ty, tz by u, v, w on

L.H.S of (1) we get

$$f(u, v, w)=t^n . f(x, y, z) \cdots \quad \cdots \quad (2)$$

where u, v, w are all functions of t,

$\therefore$ Differentiating (2) w. r. to t.

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial t}+\frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial t}+\frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial t}=nt^{n-1} . f(x, y, z)$$

or, $$x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}+z\frac{\partial f}{\partial v}=n . t^{n-1}. f(x, y, z)$$

which is true for all values of t.

$\therefore$ Putting $t=1$ we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=n . f(x, y, z).$$

3.10. Converse of Euler's Theorem for a function of three variables.

If $f(x, y, z)$ admits of continuous partial derivatives satisfying the relation

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=n. \ f(x, y, z)$$

where n is a positive integer, then

$f(x, y, z)$ is a homogeneous function of degree n.

Proof. Let $\xi=\frac{x}{z}, \eta=\frac{y}{z}, \zeta=z \quad \cdots \quad \cdots \quad (1)$

so that $x=\xi z=\xi\zeta$,

$y=\eta z=\eta\zeta$

and $z=\zeta$

Then $f(x, y, z)$ can be expressed as a function ξ, η, ζ

So let $f(x, y, z)=g(\xi, \eta, \zeta)$

Then $x\frac{\partial f}{\partial x}=x\left(\frac{\partial g}{\partial \xi}.\frac{\partial \xi}{\partial x}+\frac{\partial g}{\partial \eta}.\frac{\partial \eta}{\partial x}+\frac{\partial g}{\partial \zeta}.\frac{\partial \zeta}{\partial x}\right)$

$$=x\left(\frac{\partial g}{\partial \xi}.\frac{1}{z}+\frac{\partial g}{\partial \eta}.0+\frac{\partial g}{\partial \zeta}.0\right) \qquad \text{(from 1)}$$

$$=\frac{x}{z}\frac{\partial g}{\partial \xi}=\xi\frac{\partial g}{\partial \xi}. \qquad \cdots \quad \cdots \quad (2)$$

$$y\frac{\partial f}{\partial y}=y\left(\frac{\partial g}{\partial \xi}.\frac{\partial \xi}{\partial y}+\frac{\partial g}{\partial \eta}\frac{\partial \eta}{\partial y}+\frac{\partial g}{\partial \zeta}.\frac{\partial \zeta}{\partial y}\right)$$

$$=y\left(\frac{\partial g}{\partial \xi}.0+\frac{\partial g}{\partial \eta}\frac{1}{z}+\frac{\partial g}{\partial \eta}.0\right)$$

$$=\frac{y}{z}\frac{\partial g}{\partial \eta}=\eta\frac{\partial g}{\partial \eta} \qquad \cdots \quad \cdots \quad (3)$$

and $z\frac{\partial f}{\partial z}=z\left(\frac{\partial g}{\partial \xi}.\frac{\partial \xi}{\partial z}+\frac{\partial g}{\partial \eta}.\frac{\partial \eta}{\partial z}+\frac{\partial g}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right)$

$$=z\left\{\frac{\partial g}{\partial \xi}\left(-\frac{x}{z^2}\right)+\frac{\partial g}{\partial \eta}\left(-\frac{y}{z^2}\right)+\frac{\partial g}{\partial \zeta}.1\right\}$$

$$=-\frac{x}{z}\frac{\partial g}{\partial \xi}-\frac{y}{z}\frac{\partial g}{\partial \eta}+z\frac{\partial g}{\partial \zeta}$$

$$=-\xi\frac{\partial g}{\partial \xi}-\eta\frac{\partial g}{\partial \eta}+\zeta\frac{\partial g}{\partial \zeta}. \qquad \cdots \quad \cdots \quad (4)$$

Putting (2), (3) and (4) in the given relation

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=n.f(x, y, z)$$

we get $\zeta\frac{\partial g}{\partial \zeta}=n.g(\xi, \eta, \zeta)$

$$=n.g$$

or, $\frac{1}{g}\frac{\partial g}{\partial \zeta}=\frac{n}{\zeta}$

$\therefore$ Integrating w r. to ζ we get

$$\log g=n\log \zeta+C \text{ (constant)} \qquad \cdots \quad \cdots \quad (5)$$

Hence, the constant C is independent of ζ, but it may depend on ξ and η.

So let $C=\log \phi(\xi, \eta)$.

Then from (5)

$$\log g = n \log \zeta + \log \phi(\xi, \eta)$$

$$= \log \zeta^n . \phi(\xi, \eta)$$

$$\therefore \; g = \zeta^n \phi(\xi, \eta)$$

$$i\,e., \quad g(\xi, \eta, \zeta) = \zeta^n . \phi(\xi, \eta)$$

$$i.e., \quad f(x, y, z) = z^n . \phi\left(\frac{x}{z}, \frac{y}{z}\right).$$

which shows that $f(x, y, z)$ is a homogeneous function of degree n.

Cor. 1. If $f(x_1, x_2 \cdots x_m)$ be a homogeneous function of m variables of degree n having continuous partial derivatives, then

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \cdots + x_m \frac{\partial f}{\partial x_m} = n . f(x_1, x_2 \cdots x_m)$$

Cor. 2. If u be a homogeneous function in x, y, z of degree n, then $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ are each a homogeneous function in x, y, z of degree $(n-1)$.

We have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$

Differentiating this w.r.to x we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} + z\frac{\partial^2 u}{\partial x \partial z} = n\frac{\partial u}{\partial x}$$

$$\text{or,} \quad x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} + z\frac{\partial^2 u}{\partial x \partial z} = (n-1)\frac{\partial u}{\partial x}$$

$$\text{or,} \quad x\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right) = (n-1)\frac{\partial u}{\partial x}$$

$\therefore \; \frac{\partial u}{\partial x}$ is a homogeneous function of degree $(n-1)$.

Similarly, for $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$.

Illustrative Examples :

Ex. 1. If $v=f(u)$ where u is a homogeneous function of degree n in x, y, then $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=nu\frac{dv}{du}$.

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=x\frac{\partial}{\partial x}f(u)+y\frac{\partial}{\partial y}f(u)$$

$$=xf'(u)\frac{\partial u}{\partial x}+yf'(u)\frac{\partial u}{\partial y}$$

$$=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)f'(u)$$

$$=nu.\frac{dv}{du}.$$

Ex. 2. If $u=v(y-z,\ z-x,\ x-y)$, then prove that

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0. \qquad \textbf{(C. H. 1960)}$$

Let $\xi=y-z$, $\eta=z-x$, $\zeta=x-y$.

Then $u=v(\xi,\ \eta,\ \zeta)$

$$\therefore\quad \frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}+\frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x}$$

$$=\frac{\partial u}{\partial \zeta}\cdot 0+\frac{\partial u}{\partial \eta}\cdot(-1)+\frac{\partial u}{\partial \zeta}\cdot 1=\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}.$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}+\frac{\partial u}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial y}$$

$$=\frac{\partial u}{\partial \xi}\cdot(1)+\frac{\partial u}{\partial \eta}\cdot 0+\frac{\partial u}{\partial \zeta}(-1)=\frac{\partial u}{\partial \zeta}-\frac{\partial u}{\partial \zeta}.$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial z}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial z}+\frac{\partial u}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial z}$$

$$=\frac{\partial u}{\partial \xi}\cdot(-1)+\frac{\partial u}{\partial \eta}\cdot 1+\frac{\partial u}{\partial \zeta}\cdot 0=\frac{\partial u}{\partial \eta}-\frac{\partial u}{\partial \xi}.$$

So adding, we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$$

Ex. 3. If $u=f(x^2+2yz,\ y^2+2zx)$, prove that

$$(y^2-zx)\frac{\partial u}{\partial x}+(x^2-yz)\frac{\partial u}{\partial y}+(z^2-xy)\frac{\partial u}{\partial z}=0$$

(C. H. 1947)

Let $\xi=x^2+2yz$ and $\eta=y^2+2zx$, then $u=f(\xi,\eta)$

$$\therefore\quad \frac{\partial u}{\partial x}=\frac{\partial f}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial f}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}=\frac{\partial f}{\partial \xi}(2x)+\frac{\partial f}{\partial \eta}(2z)$$

$$\frac{\partial u}{\partial y}=\frac{\partial f}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial f}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}=\frac{\partial f}{\partial \xi}(2z)+\frac{\partial f}{\partial \eta}(2y)$$

$$\frac{\partial u}{\partial z}=\frac{\partial f}{\partial \xi}\cdot\frac{\partial \xi}{\partial z}+\frac{\partial f}{\partial \eta}\cdot\frac{\partial \eta}{\partial z}=\frac{\partial f}{\partial \xi}(2y)+\frac{\partial f}{\partial \eta}(2x)$$

$$\therefore\quad \Sigma(y^2-zx)\frac{\partial u}{\partial x}=2\frac{\partial f}{\partial \xi}\{x(y^2-zx)+z(x^2-yz)+y(z^2-xy)\}$$

$$+2\frac{\partial f}{\partial \eta}\{z(y^2-zx)+y(x^2-yz)+x(z^2-xy)\}=0.$$

Ex. 4. If $u=\tan^{-1}\dfrac{x^3+y^3}{x^2+y^2}$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\sin 2u$$

(C. H. 1963)

Here, $\tan u=\dfrac{x^3+y^3}{x^2+y^2}=\dfrac{x^3\left(1+\dfrac{y^3}{x^3}\right)}{x^2\left(1+\dfrac{y^2}{x^2}\right)}$

$$=x\left\{1+\left(\frac{y}{x}\right)^3\right\}\cdot\left\{1+\left(\frac{y}{x}\right)^2\right\}^{-}$$

$$=x\phi\left(\frac{y}{x}\right).$$

$\therefore$ $\tan u$ is of degree 1.

Hence by Euler's Theorem

$$x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u)=1\ .\ \tan u.$$

or, $x\sec^2 u\dfrac{\partial u}{\partial x}+y\sec^2 u\dfrac{\partial u}{\partial y}=\tan u$

or, $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{\tan u}{\sec^2 u}=\dfrac{1}{2}\sin 2u.$

Ex. 5. If $u(x, y, z, t)=\frac{1}{r}f(t+r)+\frac{1}{r}g(t-r)$

when $r^2=x^2+y^2+z^2$, prove that u satisfies the relation $u_{xx}+u_{yy}+u_{zz}=u_{tt}$ (C. H. 1961)

$$\frac{\partial u}{\partial x}=\frac{1}{r}f'(t+r)\frac{\partial r}{\partial x}-\frac{1}{r^2}\frac{\partial r}{\partial x}\cdot f(t+r)$$

$$+\frac{1}{r}g'(t-r)\left(-\frac{\partial r}{\partial x}\right)-\frac{1}{r^2}\frac{\partial r}{\partial x}\cdot g(t-r)$$

$$=\frac{1}{r}f'(t+r)\left(\frac{x}{r}\right)-\frac{1}{r^2}\left(\frac{x}{r}\right)f(t+r) \qquad \because \frac{\partial r}{\partial x}=\frac{x}{r}$$

$$-\frac{1}{r}g'(t-r)\left(\frac{x}{r}\right)-\frac{1}{r^2}\left(\frac{x}{r}\right)g(t-r)$$

$$=\frac{x}{r^2}f'(t+r)-\frac{x}{r^3}f(t+r)-\frac{x}{r^2}g'(t-r)-\frac{x}{r^3}g(t-r)$$

$$\therefore \frac{\partial^2 v}{\partial x}=\left\{\frac{1}{r^2}f'(t+r)-\frac{2x}{r^3}\frac{\partial r}{\partial x}f(t+r)+\frac{x}{r^2}f''(t+r)\frac{\partial r}{\partial x}\right\}$$

$$-\left\{\frac{1}{r^3}f(t+r)\quad\frac{3x}{r^4}\frac{\partial r}{\partial x}f(t+r)+\frac{x}{r^3}f'(t+r)\frac{\partial r}{\partial x}\right\}$$

$$-\left\{\frac{1}{r^2}g'(t-r)-\frac{2x}{r^3}\frac{\partial r}{\partial x}g'(t-r)+\frac{x}{r^2}g''(t-r)\left(-\frac{\partial r}{\partial x}\right)\right\}$$

$$-\left\{\frac{1}{r^3}g(t-r)-\frac{3x}{r^4}\frac{\partial r}{\partial x}g(t-r)+\frac{x}{r^3}g'(t-r)\left(-\frac{\partial r}{\partial x}\right)\right\}$$

$$=\frac{x^2}{r^5}f''(t+r)+\left(\frac{1}{r^2}-\frac{2x^2}{r^4}-\frac{x^2}{r^4}\right)f'(t+r)+\left(\frac{3x^2}{r^5}-\frac{1}{r^3}\right)f(t+r)$$

$$+\frac{x^2}{r^3}g''(t-r)+\left(\frac{2x^2}{r^4}+\frac{x^2}{r^4}-\frac{1}{r^2}\right)g'(t-r)$$

$$+\left(\frac{3x^2}{r^5}-\frac{1}{r^3}\right)g(t-r)$$

Similar expression for $\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial^2 u}{\partial z^2}$.

$$\therefore \quad u_{xx}+u_{yy}+u_{zz}=\frac{1}{r}\{f''(t+r)+g''(t-r)\}$$

$=u_{tt}$. from the given relation.

Ex. 6. If $z=xf(x+y)+yg(x+y)$ show that

$$\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0. \qquad \text{(C. H. 1971)}$$

Put $v=x+y$, Then $\frac{\partial v}{\partial x}=1$ and $\frac{\partial v}{\partial y}=1$.

Now $z=xf(v)+yg(v)$

$$\therefore \quad \frac{\partial z}{\partial x}=f(v)+xf'(v)\frac{\partial v}{\partial x}+yg'(v)\frac{\partial v}{\partial x}$$

$$=f(v)+xf'(v)+yg'(v)$$

$$\frac{\partial^2 z}{\partial x^2}=f'(v)+f'(v)+xf''(v)+yg''(v)$$

$$=2f'(v)+xf''(v)+yg''(v)$$

$$\frac{\partial z}{\partial y}=xf'(v)+g(v)+yg'(v)$$

$$\therefore \quad \frac{\partial^2 z}{\partial y^2}=xf''(v)+g'(v)+g'(v)+yg''(v)$$

$$=2g'(v)+xf''(v)+yg''(v)$$

Also $\frac{\partial^2 z}{\partial x\partial y}=f'(v)+xf''(v)+g'(v)+yg''(v)$

Hence, $\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0.$

Ex. 7. If $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$

where u is a function of x, y, z prove that

$$(u_x)^2+(u_y)^2+(u_z)^2=2(xu_x+yu_y+zu_z) \qquad \text{(C. H. 1966)}$$

Differentiating the given relation w. r. to x partially

$$\frac{2x(a^2+u)-x^2u_x}{(a^2+u)^2}-\frac{y^2u_x}{(b^2+u)^2}-\frac{z^2u_x}{(c^2+u)^2}=0$$

$$\text{or,} \quad u_x\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}=\frac{2x}{a^2+u} \quad \cdots (1)$$

Similarly, differentiating the given relation w. r. to y and z respectively, we get

$$u_y\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}=\frac{2y}{a^2+u} \quad \cdots (2)$$

and $$u_z\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}=\frac{2z}{a^2+u} \quad \cdots (3)$$

Squaring (1), (2), (3) and adding

$$\left\{(u_x)^2+(u_y)^2+(u_z)^2\right\}\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(a^2+u)^2}+\frac{z^2}{(a^2+u)^2}\right\}^2$$
$$=4\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(a^2+u)^2}+\frac{z^2}{(a^2+u)^2}\right\}$$

or, $$(u_x)^2+(u_y)^2+(u_z)^2=4\Big/\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(a^2+u)^2}+\frac{z^2}{(a^2+u)^2}\right\} \quad \cdots (4)$$

Again multiplying (1), (2), (3) by x, y, z and adding

$$(xu_x+yu_y+zu_z)\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}$$
$$=2\left\{\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right\}$$
$$=2.$$

$$\therefore \quad xu_x+yu_y+zu_z=2\Big/\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\} \quad \cdots (5)$$

Dividing (4) by (5)

$$\frac{(u_x)^2+(u_y)^2+(u_z)^2}{xu_x+yu_y+zu_z}=2.$$ Hence the result

Ex. 8. If $y_1=f_1(x_1, x_2, x_3)$, $y_2=f_2(x_1, x_2, x_3)$ are differential functions of x_1, x_2, x_3 in a region R and $x_1=\phi_1(t_1, t_2)$, $x_2=\phi_2(t_1, t_2)$, $x_2=\phi_3(t_1, t_2)$ are differential functions of t_1, t_2 in a region S, then prove that

$$\frac{\partial(y_1, y_2)}{\partial(t_1, t_2)}=\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\cdot\frac{\partial(x_1, x_2)}{\partial(t_1, t_2)}+\frac{\partial(y_1, y_2)}{\partial(x_2, x_3)}\cdot\frac{\partial(x_2, y_3)}{\partial(t_1, t_2)}$$
$$+\frac{\partial(y_1, y_2)}{\partial(x_3, x_1)}\cdot\frac{\partial(x_3, x_1)}{\partial(t_1, t_2)}$$

where $\dfrac{\partial(u, v)}{\partial(p, q)}$ denotes $\begin{vmatrix} \dfrac{\partial u}{\partial p} & \dfrac{\partial u}{\partial q} \\ \dfrac{\partial v}{\partial p} & \dfrac{\partial v}{\partial q} \end{vmatrix}$ (C. H. 1967)

we have $$\frac{\partial(y_1, y_2)}{\partial(t_1, t_2)} = \begin{vmatrix} \dfrac{\partial y_1}{\partial t_1} & \dfrac{\partial y_1}{\partial t_2} \\[2ex] \dfrac{\partial y_2}{\partial t_1} & \dfrac{\partial y_2}{\partial t_2} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1}\dfrac{\partial x_1}{\partial t_1} + \dfrac{\partial y_1}{\partial x_2}\dfrac{\partial x_2}{\partial t_1} + \dfrac{\partial y_1}{\partial x_3}\dfrac{\partial x_3}{\partial t_1} & \dfrac{\partial y_1}{\partial x_1}\dfrac{\partial x_1}{\partial t_2} + \dfrac{\partial y_1}{\partial x_2}\dfrac{\partial x_2}{\partial t_2} + \dfrac{\partial y_1}{\partial x_3}\dfrac{\partial x_3}{\partial t_2} \\[2ex] \dfrac{\partial y_2}{\partial x_1}\dfrac{\partial x_1}{\partial t_1} + \dfrac{\partial y_2}{\partial x_2}\dfrac{\partial x_2}{\partial t_1} + \dfrac{\partial y_2}{\partial x_3}\dfrac{\partial x_3}{\partial t_1} & \dfrac{\partial y_2}{\partial x_1}\dfrac{\partial x_1}{\partial t_2} + \dfrac{\partial y_2}{\partial x_2}\dfrac{\partial x_2}{\partial t_2} + \dfrac{\partial y_2}{\partial x_3}\dfrac{\partial x_3}{\partial t_2} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1}\dfrac{\partial x_1}{\partial t_1} + \dfrac{\partial y_1}{\partial x_2}\dfrac{\partial x_2}{\partial t_1}, & \dfrac{\partial y_1}{\partial x_1}\dfrac{\partial x_1}{\partial t_2} + \dfrac{\partial y_1}{\partial x_2}\dfrac{\partial x_2}{\partial t_2} \\[2ex] \dfrac{\partial y_2}{\partial x_1}\dfrac{\partial x_1}{\partial t_1} + \dfrac{\partial y_2}{\partial x_2}\dfrac{\partial x_2}{\partial t_1}, & \dfrac{\partial y_2}{\partial x_1}\dfrac{\partial x_1}{\partial t_2} + \dfrac{\partial y_2}{\partial x_2}\dfrac{\partial x_2}{\partial t_2} \end{vmatrix}$$

$$+ \begin{vmatrix} \dfrac{\partial y_1}{\partial x_2}\dfrac{\partial x_2}{\partial t_1} + \dfrac{\partial y_1}{\partial x_3}\dfrac{\partial x_3}{\partial t_1}, & \dfrac{\partial y_1}{\partial x_2}\dfrac{\partial x_2}{\partial t_2} + \dfrac{\partial y_1}{\partial x_3}\dfrac{\partial x_3}{\partial t_2} \\[2ex] \dfrac{\partial y_2}{\partial x_2}\dfrac{\partial x_2}{\partial t_1} + \dfrac{\partial y_2}{\partial x_3}\dfrac{\partial x_3}{\partial t_1}, & \dfrac{\partial y_2}{\partial x_2}\dfrac{\partial x_2}{\partial t_2} + \dfrac{\partial y_2}{\partial x_3}\dfrac{\partial x_3}{\partial t_2} \end{vmatrix}$$

$$+ \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1}\dfrac{\partial x_1}{\partial t_1} + \dfrac{\partial y_1}{\partial x_3}\dfrac{\partial x_3}{\partial t_1}, & \dfrac{\partial y_1}{\partial x_1}\dfrac{\partial x_1}{\partial t_2} + \dfrac{\partial y_1}{\partial x_3}\dfrac{\partial x_3}{\partial t_2} \\[2ex] \dfrac{\partial y_2}{\partial x_1}\dfrac{\partial x_1}{\partial t_1} + \dfrac{\partial y_2}{\partial x_3}\dfrac{\partial x_3}{\partial t_1}, & \dfrac{\partial y_2}{\partial x_1}\dfrac{\partial x_1}{\partial t_2} + \dfrac{\partial y_2}{\partial x_3}\dfrac{\partial x_3}{\partial t_2} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\[2ex] \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{vmatrix} \begin{vmatrix} \dfrac{\partial x_1}{\partial t_1} & \dfrac{\partial x_2}{\partial t_1} \\[2ex] \dfrac{\partial x_1}{\partial t_2} & \dfrac{\partial x_2}{\partial t_2} \end{vmatrix} + \begin{vmatrix} \dfrac{\partial y_1}{\partial x_2} & \dfrac{\partial y_1}{\partial x_3} \\[2ex] \dfrac{\partial y_2}{\partial x_2} & \dfrac{\partial y_2}{\partial x_3} \end{vmatrix} \begin{vmatrix} \dfrac{\partial x_2}{\partial t_1} & \dfrac{\partial x_3}{\partial t_1} \\[2ex] \dfrac{\partial x_2}{\partial t_2} & \dfrac{\partial x_3}{\partial t_2} \end{vmatrix}$$

$$+ \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_3} \\[2ex] \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_3} \end{vmatrix} \begin{vmatrix} \dfrac{\partial x_1}{\partial t_1} & \dfrac{\partial x_3}{\partial t_1} \\[2ex] \dfrac{\partial x_1}{\partial t_2} & \dfrac{\partial x_3}{\partial t_2} \end{vmatrix}$$

$$=\begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} \begin{vmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} \\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} \end{vmatrix} + \begin{vmatrix} \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \end{vmatrix} \begin{vmatrix} \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} \\ \frac{\partial x_3}{\partial t_1} & \frac{\partial x_3}{\partial t_2} \end{vmatrix}$$

$$+\begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_3} \end{vmatrix} \begin{vmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} \\ \frac{\partial x_3}{\partial t_1} & \frac{\partial x_3}{\partial t_2} \end{vmatrix}$$

$$=\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \cdot \frac{\partial(x_1, x_2)}{\partial(t_1, t_2)} + \frac{\partial(y_1, y_2)}{\partial(x_2, x_3)} \cdot \frac{\partial(x_2, x_3)}{\partial(t_1, t_2)}$$

$$+\frac{\partial(y_1, y_2)}{\partial(x_1, x_3)} \cdot \frac{\partial(x_1, x_3)}{\partial(t_1, t_2)}.$$

Ex. 9. If $f=f(x, y, z)$, $g=g(x, y, z)$ are differential functions of x, y, z and $x=x(p, q)$, $y=y(p, q)$, $z=z(p, q)$ are differential functions of p, q, then prove that

$$\frac{\partial(f, g)}{\partial(p, q)}=\frac{\partial(f, g)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(p, q)}+\frac{\partial(f, g)}{\partial(y, z)} \frac{\partial(y, z)}{\partial(p, q)}+\frac{\partial(f, g)}{\partial(z, x)} \cdot \frac{\partial(z, x)}{\partial(p, q)}$$

where $\frac{\partial(f, g)}{\partial(p, q)}$ **stands for** $f_p g_q - f_q g_p$ *(C. H. 1969)*

We have $\frac{\partial(f, g)}{\partial(p, q)}=\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix}$

$$=\begin{vmatrix} f_x x_p+f_y y_p+f_z z_p & f_x x_q+f_y y_q+f_z z_q \\ g_x x_p+g_y y_p+g_z z_p & g_x x_q+g_y y_q+g_z z_q \end{vmatrix}$$

$$=\begin{vmatrix} f_x x_p+f_y y_p, & f_x x_q+f_y y_q \\ g_x x_p+g_y y_p & g_x x_q+g_y y_q \end{vmatrix} + \begin{vmatrix} f_y y_p+f_z z_p, & f_y y_q+f_z z_q \\ g_y y_p+g_z z_p & g_y y_q+g_z z_q \end{vmatrix}$$

$$+\begin{vmatrix} f_x x_p+f_z z_p, & f_x x_q+f_z z_q \\ g_x x_p+g_z z_p, & g_x x_q+g_z z_q \end{vmatrix}$$

$$=\begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} \begin{vmatrix} x_p & y_p \\ x_q & y_q \end{vmatrix} + \begin{vmatrix} f_y & f_z \\ g_y & g_z \end{vmatrix} \begin{vmatrix} y_p & z_p \\ y_q & z_q \end{vmatrix}$$

$$+\begin{vmatrix} f_z & f_x \\ g_z & g_x \end{vmatrix} \begin{vmatrix} z_p & x_p \\ z_q & x_q \end{vmatrix}$$

$$=\begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}\begin{vmatrix} x_p & x_q \\ y_p & y_q \end{vmatrix}+\begin{vmatrix} f_y & f_z \\ g_y & g_z \end{vmatrix}\begin{vmatrix} y_p & y_q \\ z_p & z_q \end{vmatrix}$$

$$+\begin{vmatrix} f_z & f_x \\ g_z & g_x \end{vmatrix}\begin{vmatrix} z_p & z_q \\ x_p & x_q \end{vmatrix}$$

$$=\frac{\partial(f, g)}{\partial(x, y)}\cdot\frac{\partial(x, y)}{\partial(p, q)}+\frac{\partial(f, g)}{\partial(y, z)}\cdot\frac{\partial(y, z)}{\partial(p, q)}+\frac{\partial(f, g)}{\partial(z, x)}\cdot\frac{\partial(z, x)}{\partial(p, q)}.$$

Ex. 10. If Pdx+Qdy+Rdz can be made a perfect differential of some function of x, y, z, on multiplication by a factor then

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0,$$

(C. H. 1949, 54)

Let $u=f(x, y, z)$ so that

$$K(Pdx+Qdy+Rdz=du \qquad \cdots \qquad \cdots \qquad (1)$$

Then K must be some function of x, y, z.

But, since u is a function of x, y, z, we can write

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz \qquad \cdots \qquad \cdots \qquad (2)$$

Comparing (1) and (2) we get

$$\frac{\partial u}{\partial x}=KP, \qquad \cdots \qquad \cdots \qquad (3)$$

$$\frac{\partial u}{\partial y}=KQ, \qquad \cdots \qquad \cdots \qquad (4)$$

$$\frac{\partial u}{\partial z}=KR, \qquad \cdots \qquad \cdots \qquad (5)$$

Differentiating (3) w. r. to y and (4) w. r. to x.

$$\frac{\partial^2 u}{\partial y\partial x}=K\frac{\partial P}{\partial y}+P\frac{\partial K}{\partial y}$$

and $$\frac{\partial^2 u}{\partial x\partial y}=K\frac{\partial Q}{\partial x}+Q\frac{\partial K}{\partial x}.$$

Assuming $\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial^2 u}{\partial y\partial x}$, we get

$$K\frac{\partial P}{\partial y}+P\frac{\partial K}{\partial y}=K\frac{\partial Q}{\partial x}+Q\frac{\partial K}{\partial x}$$

or, $$K\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=Q\frac{\partial K}{\partial x}-P\frac{\partial K}{\partial y}. \quad \cdots \quad (6)$$

Similarly, differentiating (4) w. r to z and (5) w. r. to y

$$K\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)=R\frac{\partial K}{\partial y}-Q\frac{\partial K}{\partial z}. \quad \cdots \quad (7)$$

Also differentiating (3) w. r. to z and (5) w. r.to x

$$K\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)=P\frac{\partial K}{\partial z}-R\frac{\partial K}{\partial x}. \quad \cdots \quad (8)$$

Now multiplying (6) by R, (7) by P and (8) by Q and adding the result follows.

Exercise 3

1. If $u=\tan \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$,

   show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{1}{4}\sin 2u$.

2. If $V=f(z)$ where z is a homogeneous function in x, y of degree n, then show that

   $$xV_x+yV_y=nz\frac{dV}{dz}.$$

3. Given $f_1(x, y)=0$ and $f_2(x, z)=0$,

   show that

   $$\frac{\partial f_2}{\partial x}\cdot\frac{\partial f_1}{\partial y}\cdot\frac{dy}{dz}=\frac{\partial f_1}{\partial x}\cdot\frac{\partial f_2}{\partial z}.$$

4. If $F(p, v, t)=0$, show that

   $$\left[\frac{dp}{dt}\right]_{v\text{ const.}}\times\left[\frac{dv}{dp}\right]_{t\text{ const.}}\times\left(\frac{dt}{dv}\right)_{v\text{ const.}}=-1.$$

5. Let $z=f(x, y)$ where $x=e^u+e^{-v}$

   and $y=e^{-u}-e^v$,

   show that $\dfrac{\partial z}{\partial u}-\dfrac{\partial z}{\partial v}=x\dfrac{\partial z}{\partial x}-y\dfrac{\partial z}{\partial y}$.

6. If $u=f(x, y, z)$ possess continuous partial derivatives, and if each of x, y, z is a function of t, then prove that,

   $$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}.$$

7. V is a homogeneous function in x, y, z and t of order n; prove that
$xV_x + yV_y + zV_z + tV_t = nV$.

8. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that (C. H. 1947)
$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0.$$

9. If $xyz = \log_e u$, prove that
$$u_{xyz} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$$

10. Given $V = \log(x^3 + y^3 + z^3 - 3xyz)$
show that $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = -\dfrac{3}{(x+y+z)^2}$.

11. If $u = f(ax^2 + 2hxy + by^2)$, $v = \phi(ax^2 + 2hxy + by^2)$,
show that $\dfrac{\partial}{\partial y}\left(u\dfrac{\partial v}{\partial x}\right) - \dfrac{\partial}{\partial x}\left(u\dfrac{\partial v}{\partial y}\right)$ (C. H. 1934)

12. Let $x = x(\xi, \eta, \zeta)$, $y = (\xi, \eta, \zeta)$ are
differential function of ξ, η, ζ and $\xi = \xi(t_1\ t_2)$, $\eta = \eta(t_1\ t_2)$
and $\zeta = \zeta(t_1, t_2)$ are differential function of t_1 and t_2.
Prove that
$$\frac{\partial(x, y)}{\partial(t_1, t_2)} = \frac{\partial(x, y)}{\partial(\xi, \eta)} \cdot \frac{\partial(\xi, \eta)}{\partial(t_1, t_2)} + \frac{\partial(x, y)}{\partial(\eta, \zeta)} \cdot \frac{\partial(\eta, \zeta)}{\partial(t_1, t_2)} + \frac{\partial(x, y)}{\partial(\zeta, \xi)} \cdot \frac{\partial(\zeta, \xi)}{\partial(t_1, t_2)}$$

13. Find a function $f(x, y)$ which is a function of $x^2 + y^2$ and is also a product of the form $\Psi(x).\ \Psi(y)$. (C. H. 1962)

14. If V be a homogeneous function of x, y, z of degree n and if,
$$\log u + \frac{1}{2}(n+1)\log(x^2 + y^2 + z^2) = 0,$$
then show that
$$\frac{\partial}{\partial}\left(V\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(V\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(V\frac{\partial u}{\partial z}\right) = 0.$$

15. If $X = r\cos\theta$, $Y = r\sin\theta$, prove that
$$\frac{\partial(x, y)}{\partial(r, \theta)} = r.$$

16. Find differential coefficient of y with respect to x from the relation
$$x^y + y^x = a^b.$$ (C. H. 1944)

17. If $x^2 = vw$, $y^2 = wu$, $z^2 = uv$, and $f(x, y, z) = \phi(u, v, w)$
then show that
$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}.$$

(Maths Tripos 1933)

18. If the independent variables are changed from u, v to x, y and then to p, q, prove that

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial p} & \frac{\partial x}{\partial q} \\ \frac{\partial y}{\partial p} & \frac{\partial y}{\partial q} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial p} & \frac{\partial u}{\partial q} \\ \frac{\partial v}{\partial p} & \frac{\partial v}{\partial q} \end{vmatrix}$$ (C. H. 1953)

19. If $u=f_1(x+by)+f_2(x-by)$, prove that

$$\frac{\partial^2 u}{\partial y^2}=b^2\frac{\partial^2 u}{\partial x^2}.$$

20. Given $u=(x+y)\left\{1+f\left(\frac{y}{x}\right)\right\}$, show that

$$x\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y\partial x}\right)=y\left(\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial x\,\partial y}\right).$$

21. If $f(x, y, z)=0$, show that

$$\frac{\partial z}{\partial x}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial z}\,;\quad \frac{\partial z}{\partial y}=-\frac{\partial f}{\partial y}\Big/\frac{\partial f}{\partial z}.$$

22. Let $x=u+\frac{\sin u}{e^v}$ and $y=v+\frac{\cos u}{e^v}$, whe

u and v are both functions of x and y.

Show that $\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$.

CHAPTER IV

CHANGE OF VARIABLES

4.1. Choice of Independent Variables.

If the variables x, y, r, θ are connected by the two relations of the type

$$x=r\cos\theta \text{ and } y=r\sin\theta$$

then any one of the variables may be expressed in terms of two of the remaining three. Thus we can express x in terms of (r, θ), (r, y) or (θ, y).

Suppose we are to find $\frac{\partial r}{\partial x}$ from these relations. If x is expressed in terms of θ, y i. e., when $x=y\cos\theta$ we can not find $\frac{\partial x}{\partial r}$, since the independent variable r is absent in the equation. If x is expressed in terms of r, θ i.e., $x=r\cos\theta$, then we can find $\frac{\partial x}{\partial r}$ when θ remains constant. Again if x be expressed in terms of r, y i.e., $x=\sqrt{r^2-y^2}$ we can find $\frac{\partial x}{\partial r}$ when y remains constant. So there is no reason to believe that the value of $\frac{\partial x}{\partial r}$ when θ remains constant will be equal to the value of $\frac{\partial x}{\partial r}$ when y remains constant.

Let us now write the partial derivatives of x w. r. to r when θ and y remains constant by the notations $\left[\frac{\partial x}{\partial r}\right]_\theta$ and $\left[\frac{\partial x}{\partial r}\right]_y$

Here, it can be easily shown that

$$\left[\frac{\partial x}{\partial r}\right]_\theta \neq \left[\frac{\partial x}{\partial r}\right]_y.$$

From $x=r\cos\theta$ we get

$$\left[\frac{\partial x}{\partial r}\right]_\theta=\cos\theta. \quad \cdots \quad (1)$$

Also from $x=\sqrt{r^2-y^2}$

$$\left[\frac{\partial x}{\partial r}\right]_y=\frac{r}{\sqrt{r^2-y^2}}=\frac{r}{x}=\sec\theta \quad \cdots \quad (2)$$

Thus form (1) and (2) $\left[\frac{\partial x}{\partial r}\right]_\theta\neq\left[\frac{\partial x}{\partial r}\right]_y$.

4.2. If y be a function of x only we can always change the independent variable x by the relation $\frac{dy}{dx}=1\Big/\frac{dx}{dy}$.

But such a change is not possible if y be a function of several variables. For consider relations $x=r\cos\theta$ and $y=r\sin\theta$

Then $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$

Now from $x=r\cos\theta$

$$\left[\frac{\partial x}{\partial r}\right]_\theta=\cos\theta$$

And from $r=\sqrt{x^2+y^2}$

$$\left[\frac{\partial r}{\partial x}\right]_y=\frac{x}{\sqrt{x^2+y^2}}=\frac{x}{r}=\cos\theta$$

$$\therefore\quad \left[\frac{\partial x}{\partial r}\right]_\theta\neq 1\Big/\left[\frac{\partial r}{\partial x}\right]_y. \quad \cdots \quad (1)$$

Again from $x=r\cos\theta$

$$\left[\frac{\partial x}{\partial \theta}\right]_r=-r\sin\theta$$

And from $\theta=\tan^{-1}\left(\frac{y}{x}\right)$

$$\left[\frac{\partial \theta}{\partial x}\right]_y=-\frac{\frac{y}{x^2}}{1+\frac{y^2}{x^2}}=-\frac{y}{x^2+y^2}=-\frac{y}{r^2}=-\frac{\sin\theta}{r}$$

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From $x = r\cos\theta$ we get

$$\left[\frac{\partial x}{\partial r}\right]_\theta = \cos\theta. \qquad \cdots \qquad (1)$$

Also from $x = \sqrt{r^2 - y^2}$

$$\left[\frac{\partial x}{\partial r}\right]_y = \frac{r}{\sqrt{r^2 - y^2}} = \frac{r}{x} = \sec\theta \qquad \cdots \qquad (2)$$

Thus form (1) and (2) $\left[\frac{\partial x}{\partial r}\right]_\theta \neq \left[\frac{\partial x}{\partial r}\right]_y$.

4.2. If y be a function of x only we can always change the independent variable x by the relation $\frac{dy}{dx} = 1\Big/\frac{dx}{dy}$.

But such a change is not possible if y be a function of several variables. For consider relations $x = r\cos\theta$ and $y = r\sin\theta$

Then $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Now from $x = r\cos\theta$

$$\left[\frac{\partial x}{\partial r}\right]_\theta = \cos\theta$$

And from $r = \sqrt{x^2 + y^2}$

$$\left[\frac{\partial r}{\partial x}\right]_y = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r} = \cos\theta$$

$$\therefore \quad \left[\frac{\partial x}{\partial r}\right]_\theta \neq 1\Big/\left[\frac{\partial r}{\partial x}\right]_y. \qquad \cdots \qquad (1)$$

Again from $x = r\cos\theta$

$$\left[\frac{\partial x}{\partial \theta}\right]_r = -r\sin\theta$$

And from $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

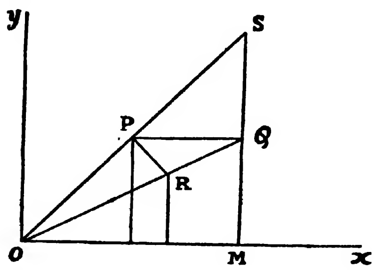
$$\left[\frac{\partial \theta}{\partial x}\right]_y = -\frac{\frac{y}{x^2}}{1 + \frac{y^2}{x^2}} = -\frac{y}{x^2 + y^2} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}$$

$$\therefore \quad \left[\frac{\partial x}{\partial \theta}\right]_r \neq 1 \Big/ \left[\frac{\partial \theta}{\partial x}\right]_y \quad \ldots \quad (2)$$

In case (1) $\frac{\partial x}{\partial r}$ is the partial differentiation of x, w. r. to r when θ remains constant and $\frac{\partial r}{\partial x}$ is the partial differentiation of r, w. r. to x when y remains constant, so there is no reason that *L. H. S.* should be equal to the *R. H. S.* Similar reasons for (2).

Geometrical interpretaton for inequality

Let P be a point whose cartesian co-ordinates are (x, y) with respect to OX and OY as axes of co-ordinates. Let (r, θ) be its polar co-ordinates.



Let PQ $(=\delta x)$ be the increment of x when y remains constant. Then PQ is parallel to OX.

Join OQ and take a point R in OQ such that $OR=OP$. Then $OR=r$ and RQ $(=\delta r)$ is the increment of r corresponding to the increment PQ $(=\delta x)$ of x.

$$\therefore \quad \frac{\partial x}{\partial r} = Lt\frac{\delta x}{\delta r} = Lt\frac{PQ}{RQ}, \text{ or, } \frac{\partial r}{\partial x} = Lt\frac{RQ}{PQ}.$$

Again let PS $(=\delta r)$ be the increment of r when θ remains constant. Then $\frac{\partial x}{\partial r} = Lt\frac{\partial x}{\partial r} = Lt\frac{PQ}{PS}$.

But $RQ \neq PS$

$$\therefore \quad \left(\frac{\partial x}{\partial r}\right)_y \neq \left(\frac{\partial x}{\partial r}\right)_\theta$$

$$\text{And} \quad \left(\frac{\partial x}{\partial r}\right)_y \neq 1 \Big/ \left(\frac{\partial r}{\partial x}\right)_\theta.$$

4.3. For a function of several variables, the independent variables can be changed to a second set of variables by proper substitution. This is illustrated in the following examples :

Ex. 1. If v be a function of r alone where $\mathbf{r^2=x^2+y^2}$, show that

$$\frac{\partial^2 \mathbf{v}}{\partial \mathbf{x}^2}+\frac{\partial^2 \mathbf{v}}{\partial \mathbf{y}^2}=\frac{\partial^2 \mathbf{v}}{\partial \mathbf{r}^2}+\frac{1}{\mathbf{r}}\frac{\partial \mathbf{v}}{\partial \mathbf{r}}$$

Since $v=f(r)$, $\dfrac{dv}{dr}=f'(r)$

Again since r is a function of x, y

$$\frac{\partial v}{\partial x}=f'(r)\frac{\partial r}{\partial x}=\frac{dv}{dr}\cdot\frac{x}{r}$$

$$\therefore\quad \frac{\partial}{\partial x}=\frac{x}{r}\cdot\frac{d}{dr} \qquad \ldots \qquad \ldots \qquad (1)$$

$$\therefore\quad \frac{\partial^2 v}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}\left\{\frac{x}{r}\frac{dv}{dr}\right\}$$

$$=\frac{1}{r}\frac{dv}{dr}+x.\frac{\partial}{\partial x}\left\{\frac{1}{r}\frac{dv}{dr}\right\}$$

$$=\frac{1}{r}\frac{dv}{dr}+x.\frac{x}{r}\frac{d}{dr}\left\{\frac{1}{r}\frac{dv}{dr}\right\} \quad \text{by (1)}$$

$$=\frac{1}{r}\frac{dv}{dr}+\frac{x^2}{r}\left\{-\frac{1}{r^2}\frac{dv}{dr}+\frac{1}{r}\frac{d^2v}{dr^2}\right\}$$

$$=\frac{1}{r}\frac{dv}{dr}-\frac{x^2}{r^3}\frac{dv}{dr}+\frac{x^2}{r^2}\frac{d^2v}{dr^2}.$$

Similarly, $\dfrac{\partial^2 v}{\partial y^2}=\dfrac{1}{r}\dfrac{dv}{dr}-\dfrac{y^2}{r^3}\dfrac{dv}{dr}+\dfrac{y^2}{r^2}\dfrac{d^2v}{dr^2}.$

$$\cdot\quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{2}{r}\frac{dv}{dr}-\frac{(x^2+y^2)}{r^3}\frac{dv}{dr}+\frac{x^2+y^2}{r^2}\frac{d^2v}{dr^2}$$

$$=\frac{2}{r}\frac{dv}{dr}-\frac{1}{r}\frac{dv}{dr}+\frac{d^2v}{dr^2}$$

$$=\frac{1}{r}\frac{dv}{dr}+\frac{d^2v}{dr^2}.$$

Ex. 2. If v is a function of r alone where $r^2 = x^2 + y^2 + z^2$, show that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{d^2 v}{dr^2} + \frac{2}{r}\frac{dv}{dr}.$$

Since v is a function of r alone

$$\frac{\partial v}{\partial x} = \frac{dv}{dr}\frac{\partial r}{\partial x} = \frac{x}{r}\frac{dv}{dr} \qquad \text{(see Ex. 1)}$$

$$\therefore \quad \frac{\partial}{\partial x} = \frac{x}{r}\frac{d}{dr}.$$

$$\therefore \quad \frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r}\frac{dv}{dr}\right)$$

$$= \frac{1}{r}\frac{dv}{dr} + x\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{dv}{dr}\right)$$

$$= \frac{1}{r}\frac{dv}{dr} + x.\frac{x}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{dv}{dr}\right)$$

$$= \frac{1}{r}\frac{dv}{dr} + \frac{x^2}{r}\left(\frac{1}{r}\frac{d^2 v}{dr^2} - \frac{1}{r^2}\frac{dv}{dr}\right)$$

$$= \frac{1}{r}\frac{dv}{dr} + \frac{x^2}{r^2}\frac{d^2 v}{dr^2} - \frac{x^2}{r^3}\frac{dv}{dr}.$$

Similarly, $\dfrac{\partial^2 v}{\partial y^2} = \dfrac{1}{r}\dfrac{dv}{dr} + \dfrac{y^2}{r^2}\dfrac{d^2 v}{dr^2} - \dfrac{y^2}{r^3}\dfrac{dv}{dr}$

And $\dfrac{\partial^2 v}{\partial z^2} = \dfrac{1}{r}\dfrac{dv}{dr} + \dfrac{z^2}{r^2}\dfrac{d^2 v}{dr^2} - \dfrac{z^2}{r^3}\dfrac{dv}{dr}.$

$$\therefore \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{3}{r}\frac{dv}{dr} + \frac{x^2+y^2+z^2}{r^2}\frac{d^2 v}{dr^2} - \frac{x^2+y^2+z^2}{r^3}\frac{dv}{dr}$$

$$= \frac{3}{r}\frac{dv}{dr} + \frac{d^2 v}{dr^2} - \frac{1}{r}\frac{dv}{dr}$$

$$= \frac{d^2 v}{dr^2} + \frac{2}{r}\frac{dv}{dr}.$$

Ex. 3. If $x = r\cos\theta$, $y = r\sin\theta$ and v is a function of x, y possessing partial derivations of the 2nd order

prove that $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = \dfrac{1}{r^2}\left\{r\dfrac{\partial}{\partial r}\left(r\dfrac{\partial v}{\partial r}\right) + \dfrac{\partial^2 v}{\partial \theta^2}\right\}$

Since, x, y are functions of (r, θ), v is a function of (r, θ). So we write $v=f(r, \theta)$

$$\therefore \quad \frac{\partial v}{\partial x}=\frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$=\frac{\partial v}{\partial r}\cdot\frac{x}{r}-\frac{\partial v}{\partial \theta}\cdot\frac{y}{r^2}$$

$$=\frac{\partial v}{\partial r}\cos\theta-\frac{\partial v}{\partial \theta}\cdot\frac{\sin\theta}{r} \quad \cdots \quad (1)$$

$$\left[\because \quad r^2=x^2+y^2 \quad \frac{\partial r}{\partial x}=x/r\right.$$

$$\left.\because \quad \theta=\tan^{-1}\frac{y}{x}, \quad \frac{\partial \theta}{\partial x}=-\frac{y}{r^2}\right]$$

$$\frac{\partial v}{\partial y}=\frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$=\frac{\partial v}{\partial r}\cdot\frac{y}{r}+\frac{\partial v}{\partial \theta}\cdot\frac{x}{r^2}$$

$$\left[\because \quad \theta=\tan^{-1}\frac{y}{x}, \quad \frac{\partial \theta}{\partial y}=\frac{x}{r^2}\right]$$

$$=\frac{\partial v}{\partial r}\sin\theta+\frac{\partial v}{\partial \theta}\cdot\frac{\cos\theta}{r} \quad \cdots \quad \cdots \quad (2)$$

Squaring (1), (2) and adding

$$\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2=\left(\frac{\partial v}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial v}{\partial \theta}\right)^2$$

Now differentiating (1) w. r. to x

$$\frac{\partial^2 v}{\partial x^2}=\frac{\partial^2 v}{\partial r^2}\,\frac{\partial r}{\partial x}\cos\theta+\frac{\partial^2 v}{\partial \theta\partial r}\cdot\frac{\partial \theta}{\partial x}\cdot\cos\theta-\frac{\partial v}{\partial r}\sin\theta\,\frac{\partial \theta}{\partial x}$$

$$-\frac{\partial^2 v}{\partial \theta^2}\cdot\frac{\partial \theta}{\partial x}\cdot\frac{\sin\theta}{r}-\frac{\partial^2 v}{\partial r\partial \theta}\,\frac{\partial r}{\partial \theta}\cdot\frac{\sin\theta}{r}-\frac{\partial v}{\partial \theta}\cdot\frac{\cos\theta}{r}\frac{\partial \theta}{\partial x}$$

$$+\frac{\partial v}{\partial \theta}\,\frac{\sin\theta}{r^2}\,\frac{\partial r}{\partial x}$$

$$=\frac{\partial^2 v}{\partial r^2}\cos^2\theta-\frac{\partial^2 v}{\partial\theta\partial r}\ \frac{\sin\theta\cos\theta}{r}+\frac{\partial v}{\partial r}\ \frac{\sin^2\theta}{r}$$

$$+\frac{\partial^2 v}{\partial\theta^2}\ \frac{\sin\theta}{r}\cdot\frac{y}{r^2}-\frac{\partial^2 v}{\partial r.\partial\theta}\ \frac{\sin\theta\cos\theta}{r}+\frac{\partial v}{\partial\theta}\cdot\ \frac{\sin\theta\cos\theta}{r^2}$$

$$+\frac{\partial v}{\partial\theta}\ \frac{\sin\theta\cos\theta}{r^2}.$$

$$=\frac{\partial^2 v}{\partial r^2}\cos^2\theta+\frac{\partial^2 v}{\partial\theta^2}\ \frac{\sin^2\theta}{r^2}-2\frac{\partial^2 v}{\partial r\partial\theta}\cdot\ \frac{\sin\theta\cos\theta}{r}$$

$$+\frac{\partial v}{\partial r}\ \frac{\sin^2\theta}{r}+2\frac{\partial v}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2}$$

Similarly, $\dfrac{\partial^2 v}{\partial y^2}=\dfrac{\partial^2 v}{\partial r^2}\sin^2\theta+\dfrac{\partial^2 v}{\partial\theta^2}\ \dfrac{\cos^2\theta}{r^2}+2\dfrac{\partial^2 v}{\partial r\,\partial\theta}\dfrac{\cos\theta\sin\theta}{r}$

$$+\frac{\partial v}{\partial r}\ \frac{\cos^2\theta}{r}-2\frac{\partial v}{\partial\theta}\ \frac{\cos\theta\sin\theta}{r^2}$$

$$\therefore\quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 v}{\partial r^2}+\frac{1}{r^2}\ \frac{\partial^2 v}{\partial\theta^2}+\frac{1}{r}\ \frac{\partial v}{\partial r}\quad [C.H.\ 1964\ (old)]$$

$$=\frac{1}{r^2}\left\{r\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)\right\}+\frac{1}{r^2}\ \frac{\partial^2 v}{\partial\theta^2}$$

$$=\frac{1}{r^2}\left\{r\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{\partial^2 v}{\partial\theta^2}\right\}$$

Ex. 4. Let $u=f(x, y)$ be a function of two independent variables x, y where $x=\xi\cos\alpha-\eta\sin\alpha$, $y=\xi\sin\alpha+\eta\cos\alpha$.

Prove that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial^2 u}{\partial\xi^2}+\dfrac{\partial^2 u}{\partial\eta^2}$. (C. H. 1968)

$u=f(x, y)$ also $x=f_1(\xi, \eta)$, $y=f_2(\xi, \eta)$

$$\therefore\quad u=\phi(\xi, \eta)$$

$$\therefore\quad \frac{\partial u}{\partial\xi}=\frac{\partial u}{\partial x}\cdot\ \frac{\partial x}{\partial\xi}+\frac{\partial u}{\partial y}\ \frac{\partial y}{\partial\xi}$$

$$=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\sin\alpha\quad \cdots\quad \cdots\quad (1)$$

$$\therefore\quad \frac{\partial u}{\partial\eta}=\frac{\partial u}{\partial x}\ \frac{\partial x}{\partial\eta}+\frac{\partial u}{\partial y}\ \frac{\partial y}{\partial\eta}$$

$$=-\sin\alpha\,\frac{\partial u}{\partial x}+\cos\alpha\,\frac{\partial u}{\partial y}.\quad \cdots\quad (2)$$

Differentiating (1) w. r. to ξ

$$\frac{\partial^2 u}{\partial \xi^2}=\cos\alpha\ \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right)+\sin\alpha\ .\ \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial y}\right)$$

$$=\cos\alpha\left(\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial^2 u}{\partial y\partial x}\cdot\frac{\partial y}{\partial \xi}\right)$$

$$+\sin\alpha\left(\frac{\partial^2 u}{\partial x\partial y}\ \frac{\partial x}{\partial \xi}+\frac{\partial^2 u}{\partial y^2}\ \frac{\partial y}{\partial \xi}\right)$$

$$=\cos\alpha\left(\cos\alpha\ \frac{\partial^2 u}{\partial x^2}+\sin\alpha\frac{\partial^2 u}{\partial y\partial x}\right)$$

$$+\sin\alpha\left(\cos\alpha\ \frac{\partial^2 u}{\partial x\partial y}+\sin\alpha\ \frac{\partial^2 u}{\partial y^2}\right)\cdots\quad(3)$$

Similarly, differentiating (2) w. r. to η

$$\frac{\partial^2 u}{\partial \eta^2}=-\sin\alpha\left(-\sin\alpha\ \frac{\partial^2 u}{\partial x^2}+\cos\alpha\ \frac{\partial^2 u}{\partial y\partial x}\right)$$

$$+\cos\alpha\left(-\sin\alpha\ \frac{\partial^2 u}{\partial x\partial y}+\cos\alpha\ \frac{\partial^2 u}{\partial y^2}\right)\quad\cdots\quad(4)$$

Adding (3) and (4)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}.$$

Ex. 5. If $\mathbf{x+y=(u+v)^3}$, $\mathbf{x-y=(u-v)^3}$, show that

$$9\left(x^2-y^2\right)\left(\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}\right)=\left(u^2-v^2\right)\left(\frac{\partial^2 f}{\partial u^2}-\frac{\partial^2 f}{\partial v^2}\right)$$

It being given that f(x, y) has continuous partial derivatives of the first two orders.

We have $x=u^3+3uv^2$ and $y=v^3+3u^2v$

Now $$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$=3(u^2+v^2)\frac{\partial f}{\partial x}+6uv\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}.$$

$$=6uv\frac{\partial f}{\partial x}+3(u^2+v^2)\frac{\partial f}{\partial y}.$$

$$\therefore \quad \frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}=3(u+v)^2\frac{\partial f}{\partial x}+3(u+v)^2\frac{\partial f}{\partial y}$$

and $$\frac{\partial f}{\partial u}-\frac{\partial f}{\partial v}=3(u-v)^2\frac{\partial f}{\partial x}-3(u-v)^2\frac{\partial f}{\partial y}$$

So we have the operator

$$\frac{\partial}{\partial u}+\frac{\partial}{\partial y}=3(u+v)^2\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)$$

and $$\frac{\partial}{\partial u}-\frac{\partial}{\partial y}=3(u-v)^2\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right).$$

$$\therefore \quad (u+v)\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)=3(u+v)^3\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)$$

$$=3(x+y)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)$$

and $$(u-v)\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial v}\right)=3(u-v)^3\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)$$

$$=3(x-y)\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)$$

$$\therefore \quad (u+v)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right).(u-v)\left(\frac{\partial f}{\partial u}-\frac{\partial f}{\partial v}\right)$$

$$=3(x+y)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left\{3(x-y)\left(\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\right)\right\}$$

$$\because \quad \left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)(u-v)=0.$$

or, $$(u^2-v^2)\left(\frac{\partial^2 f}{\partial u^2}-\frac{\partial^2 f}{\partial v^2}\right)=9(x^2-y^2)\left(\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}\right).$$

Ex. 6. If u be a function of x, y having continuous partial derivatives up to the 2nd order and the variables x, y are changed to ξ, η by the transformation

$$(\mathbf{x}+\mathbf{y})=(\xi+\eta)^n,\ (\mathbf{x}-\mathbf{y})=(\xi-\eta)^n$$

then prove that

$$(\mathbf{x}^2-\mathbf{y}^2)\left\{\frac{\partial^2\mathbf{u}}{\partial\mathbf{x}^2}-\frac{\partial^2\mathbf{u}}{\partial\mathbf{y}^2}\right\}=\frac{1}{n^2}(\xi^2-\eta^2)\left\{\frac{\partial^2\mathbf{u}}{\partial\xi^2}+\frac{\partial^2\mathbf{u}}{\partial\eta^2}\right\}.$$

[*C. H. 1962*]

Solving the given relations

$$x=\frac{1}{2}\{(\xi+\eta)^n+(\xi-\eta)^n\},\quad y=\frac{1}{2}\{(\xi+\eta)^n-(\xi-\eta)^n\}.$$

$\therefore\quad \dfrac{\partial x}{\partial \xi}=\dfrac{n}{2}\{(\xi+\eta)^{n-1}+(\xi-\eta)^{n-1}\}$

and $\quad \dfrac{\partial x}{\partial \eta}=\dfrac{n}{2}\{(\xi+\eta)^{n-1}-(\xi-\eta)^{n-1}\}$

$\therefore\quad \dfrac{\partial^2 x}{\partial \xi^2}=\dfrac{n(n-1)}{2}\{(\xi+\eta)^{n-2}+(\xi-\eta)^{n-2}\}\quad \cdots \quad (1)$

and $\quad \dfrac{\partial^2 x}{\partial \eta^2}=\dfrac{n(n-1)}{2}\{(\xi+\eta)^{n-2}+(\xi-\eta)^{n-2}\}\quad \cdots \quad (2)$

So from (1) and (2) $\quad \dfrac{\partial^2 x}{\partial \xi^2}=\dfrac{\partial^2 x}{\partial \eta^2}\quad \cdots \quad \cdots \quad (3)$

Also $\quad \dfrac{\partial y}{\partial \xi}=\dfrac{n}{2}\{(\xi+\eta)^{n-1}-(\xi-\eta)^{n-1}\}$

and $\quad \dfrac{\partial y}{\partial \eta}=\dfrac{n}{2}\{(\xi+\eta)^{n-1}+(\xi-\eta)^{n-1}\}$

$\therefore\quad \dfrac{\partial^2 y}{\partial \xi^2}=\dfrac{n(n-1)}{2}\{(\xi+\eta)^{n-2}-(\xi-\eta)^{n-2}\}\quad \cdots \quad (4)$

and $\quad \dfrac{\partial^2 y}{\partial \eta^2}=\dfrac{n(n-1)}{2}\{(\xi+\eta)^{n-2}-(\xi-\eta)^{n-2}\}\quad \cdots \quad (5)$

So from (4) and (5) $\quad \dfrac{\partial^2 y}{\partial \xi^2}=\dfrac{\partial^2 y}{\partial \eta^2}\quad \cdots \quad \cdots \quad (6)$

Now $\quad \dfrac{\partial u}{\partial \xi}=\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial \xi}+\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial \xi}$

and $\quad \dfrac{\partial u}{\partial \eta}=\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial \eta}+\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial \eta}.$

$$\therefore\quad \frac{\partial^2 u}{\partial \xi^2}=\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial x}{\partial \xi}\right)^2+\frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial \xi^2}+\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial y}{\partial \xi}\right)^2+\frac{\partial u}{\partial y}\frac{\partial^2 y}{\partial \xi^2}$$

$$\frac{\partial^2 u}{\partial \eta^2}=\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial x}{\partial \eta}\right)^2+\frac{\partial u}{\partial x}\cdot\frac{\partial^2 x}{\partial \eta^2}+\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial y}{\partial \eta}\right)^2+\frac{\partial u}{\partial y}\frac{\partial^2 y}{\partial \eta^2}$$

$$\therefore\quad \frac{\partial^2 u}{\partial \xi^2}-\frac{\partial^2 u}{\partial \eta^2}=\frac{\partial^2 u}{\partial x^2}\left\{\left(\frac{\partial x}{\partial \xi}\right)^2-\left(\frac{\partial x}{\partial \eta}\right)^2\right\}+\frac{\partial^2 u}{\partial y^2}\left\{\left(\frac{\partial y}{\partial \xi}\right)^2-\left(\frac{\partial y}{\partial \eta}\right)^2\right\}$$

using (3) and (6)

$$=\frac{\partial^2 u}{\partial x^2}.n^2(\xi+\eta)^{n-1}.(\xi-\eta)^{n-1}-\frac{\partial^2 u}{\partial y^2}.n^2(\xi+\eta)^{n-1}.(\xi-\eta)^{n-1}$$

$$=\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\right).n^2(\xi+\eta)^{n-1}.(\xi-\eta)^{n-1}$$

$$=\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\right).n^2.\frac{(\xi+\eta)^n}{(\xi+\eta)}.\frac{(\xi-\eta)^n}{(\xi-\eta)}$$

$$=\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\right).n^2\frac{(x+y)(x-y)}{\xi^2-\eta^2}$$

or, $$\frac{1}{n^2}(\xi^2-\eta^2)\left\{\frac{\partial^2 u}{\partial \xi^2}-\frac{\partial^2 u}{\partial \eta^2}\right\}=(x^2-y^2)\left\{\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\right\}.$$

Ex. 7. If $x=c\cosh\xi\cos\eta$, $y=c\sinh\xi\sin\eta$, show that

$$\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}=\tfrac{1}{2}c^2[\cosh 2\xi-\cos 2\eta]\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right).$$

[C. H. 1965 (old)

Since $x=c\cosh\xi\cos\eta$, $\quad y=c\sinh\xi\sin\eta$

$$\frac{\partial x}{\partial \xi}=c\sinh\xi\cos\eta,\qquad \frac{\partial y}{\partial \xi}=c\cosh\xi.\sin\eta.$$

Now $$\frac{\partial u}{\partial \xi}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial \xi}$$

$$=c\sinh\xi\cos\eta\frac{\partial u}{\partial x}+c\cosh\xi.\sin\eta\frac{\partial u}{\partial y}.$$

$$\therefore\quad \frac{\partial^2 u}{\partial \xi^2}=c\cos\eta\frac{\partial}{\partial \xi}\left(\sinh\xi.\frac{\partial u}{\partial x}\right)+c\sin\eta\frac{\partial}{\partial \xi}\left(\cosh\xi.\frac{\partial u}{\partial y}\right)$$

$$=c.\cos\eta\cosh\xi\frac{\partial u}{\partial x}+c\cos\eta\sinh\xi$$

$$\left(\frac{\partial^2 u}{\partial x^2}.\frac{\partial x}{\partial \xi}+\frac{\partial^2 u}{\partial y\partial x}.\frac{\partial y}{\partial \xi}\right)+c\sin\eta\sinh\xi\frac{\partial u}{\partial y}$$

$$+c\sin\eta\cosh\xi\left(\frac{\partial^2 u}{\partial x\partial y}.\frac{\partial x}{\partial \xi}+\frac{\partial^2 u}{\partial y^2}.\frac{\partial y}{\partial \xi}\right)$$

$$=c\cos\eta\cosh\xi\frac{\partial u}{\partial x}+c\cos\eta\sinh\xi$$

$$\left(c\sinh\xi.\cos\eta.\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y\partial x}.c\cosh\xi\sin\right.$$

$$+c\sin\eta\sinh\xi\frac{\partial u}{\partial x}+c\sin\eta\cosh\xi\left(c\sinh\xi\cos\eta\right.$$

$$\left.\times\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}c\cosh\xi\sin\eta\right). \quad \cdots \quad (1)$$

Again $\dfrac{\partial x}{\partial\eta}=-c\cosh\xi\sin\eta$ and $\dfrac{\partial y}{\partial\eta}=c\sinh\xi\cos\eta$.

$$\therefore\quad \frac{\partial u}{\partial\eta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial\eta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\eta}.$$

$$=-c\cosh\xi\sin\eta\frac{\partial u}{\partial x}+c\sinh\xi\cos\eta\frac{\partial u}{\partial y}.$$

or, $$\frac{\partial^2 u}{\partial\eta^2}=-c\cosh\xi\frac{\partial}{\partial\eta}\left(\sin\eta\frac{\partial u}{\partial x}\right)+c\sinh\xi\frac{\partial}{\partial\eta}\left(\cos\eta\frac{\partial u}{\partial y}\right)$$

$$=-c\cosh\xi\cos\eta\frac{\partial u}{\partial x}-c\cosh\xi\,.\sin\eta$$

$$\times\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial\eta}+\frac{\partial^2 u}{\partial y\partial x}\frac{\partial y}{\partial\eta}\right)$$

$$-c\sinh\xi\sin\eta\frac{\partial u}{\partial y}+c\sinh\xi\cos\eta\left(\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial\eta}+\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial\eta}\right)$$

$$=-c\cosh\xi.\cos\eta\frac{\partial u}{\partial x}-c\cosh\xi\sin\eta\left\{\frac{\partial^2 u}{\partial x^2}(-c\cosh\xi\right.$$

$$\left.\sin\eta)+\frac{\partial^2 u}{\partial y\partial x}c\sinh\xi\cos\eta\right\}-c\sinh\xi\sin\eta\frac{\partial u}{\partial y}$$

$$+c\sinh\xi\cos\eta\left\{\frac{\partial^2 u}{\partial x\partial y}(-c\cosh\xi.\sin\eta)\right.$$

$$\left.+\frac{\partial^2 u}{\partial y^2}(c\sinh\xi\cos\eta)\right\} \quad \cdots \quad (2)$$

Adding (and (2)

$$\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}=c^2\cos^2\eta\sinh^2\xi\frac{\partial^2 u}{\partial x^2}+c^2\cosh^2\xi\sin^2\eta\frac{\partial^2 u}{\partial x^2}$$

$$+c^2\sin^2\eta.\cosh^2\xi\frac{\partial^2 u}{\partial y^2}+c^2\sinh^2\xi\cos^2\eta\frac{\partial^2 u}{\partial y^2}.$$

$$=c^2\cos^2\eta.\sinh^2\xi\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right\}$$

$$+c^2\sin^2\eta.\cosh^2\xi\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right\}$$

$$=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)(\cos^2\eta\sinh^2\xi+\sin^2\eta\cosh^2\xi)$$

$$=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\{(1-\sin^2\eta)\sinh^2\xi+\sin^2\eta\cosh^2\xi\}$$

$$=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\{\sinh^2\xi+\sin^2\eta(\cosh^2\xi-\sinh^2\xi)\}$$

$$=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)(\sinh^2\xi+\sin^2\eta)$$

$$=\frac{1}{2}c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\{(\cosh 2\xi-1)+(1-\cos 2\eta)\}$$

$$=\frac{1}{2}c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)(\cosh 2\xi-\cos 2\eta).$$

Ex. 8. If f(x, y, z) be a homogeneous function of x, y, z of degree n which satisfies Laplaces equation

$$\Delta^2\mathbf{f}=\frac{\partial^2\mathbf{f}}{\partial\mathbf{x}^2}+\frac{\partial^2\mathbf{f}}{\partial\mathbf{y}^2}+\frac{\partial^2\mathbf{f}}{\partial\mathbf{z}^2}=0,$$

prove that $\Delta^2(\mathbf{r}^{2m}.\mathbf{f})=2\mathbf{m}(2\mathbf{n}+2\mathbf{m}+1)\mathbf{r}^{2m-2}.\mathbf{f}$ where

$$\mathbf{r}^2=\mathbf{x}^2+\mathbf{y}^2+\mathbf{z}^2.$$ *(C. H. 1962)*

Since $r^2=x^2+y^2+z^2$, $\frac{\partial r}{\partial x}=\frac{x}{r}$, $\frac{\partial r}{\partial y}=\frac{y}{r}$, $\frac{\partial r}{\partial z}=\frac{z}{r}$.

$$\therefore\quad \frac{\partial}{\partial x}(r^{2m}.f)=2mr^{2m-1}\frac{\partial r}{\partial x}.f+r^{2m}\frac{\partial f}{\partial x}.$$

$$=2mr^{2m-1}.\frac{x}{r}.f+r^{2m}.f_x$$

$$=2mr^{2m-2}.x.f+r^{2m}.f_x.$$

$$\therefore\quad \frac{\partial^2}{\partial x^2}(r^{2m}.f)=2m(2m-2)r^{2m-3}\frac{\partial r}{\partial x}x.f+2mr^{2m-2}.f$$

$$+2mr^{2m-2}x\frac{\partial f}{\partial x}+2mr^{2m-1}\frac{\partial r}{\partial x}f_x+r^{2m}.f_{xx}$$

$$=2m(2m-2)r^{2m-4}.x^2f+2mr^{2m-2}.f+4mr^{2m-2}.xf_x$$
$$+r^{2m}f_{xx}\qquad\cdots\qquad\cdots\qquad(1)$$

Similarly,

$$\frac{\partial^2}{\partial y^2}(r^{2m}.\ f)=2m(2m-2)r^{2m-4}.\ y^2f+2mr^{2m-2}.\ f$$

$$+4mr^{2m-2}.\ yf_y+r^{2m}f_{yy} \quad \cdots \quad \cdots \quad (2)$$

$$\frac{\partial^2}{\partial z^2}(r^{2m}.\ f)=2m(2m-2)r^{2m-4}.\ z^2f+2mr^{2m-2}.\ f$$

$$+4mr^{2m-2}.\ zf_z+r^{2m}.\ f_{zz} \quad \cdots \quad (3)$$

Adding (1), (2) and (3)

$$\Delta^2(r^{2m}.\ f)=2m(2m-2)r^{2m-4}(x^2+y^2+z^2)f+6mr^{2m-2}.\ f$$

$$+4mr^{2m-2}(xf_x+yf_y+zf_z)+r^{2m}(f_{xx}+f_{yy}+f_{zz}).$$

$$=2m(2m-2)r^{2m-4}.\ r^2f+6mr^{2m-2}.\ f+4mr^{2m-2}.\ nf$$

$$[\ \therefore\quad \Sigma f_{xx}=0\ ]$$

$$=2m(2m+2n+1)r^{2m-2}.\ f.$$

Ex. 9. If v_n is a homogeneous functions of x, y, z of degree n and $r^2=x^2+y^2+z^2$, then

$$\nabla^2(r^m v_n)=m(m+2n+1)r^{m-2}.\ v_n+r^m\nabla^2.\ v_n \quad \text{where}$$

$$\nabla^2=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$$

Deduce that if further v_n satisties the equation $\nabla^2 v_n=0$, so does $r^{-2n-1}.\ v_n$. (C. H. 1967)

Since, $r^2=x^2+y^2+z^2$,

$$\frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}.$$

Now $\dfrac{\partial}{\partial x}(r^m v_n)=mr^{m-1}\dfrac{\partial r}{\partial x}.\ v_n+r^m\dfrac{\partial v_n}{\partial x}$

$$=mxr^{m-2}.\ v_n+r^m\frac{\partial v_n}{\partial x}.$$

$$\therefore\quad \frac{\partial^2}{\partial x^2}(r^m.\ v_n)=m(m-2)r^{m-3}\frac{\partial r}{\partial x}.x.v_n+mr^{m-2}v_n$$

$$+mxr^{m-2}\frac{\partial v_n}{\partial x}+mr^{m-1}\frac{\partial r}{\partial x}.\frac{\partial v_n}{\partial x}+r^m\frac{\partial^2 v_n}{\partial x^2}$$

$$=m(m-2)x^2r^{m-4}\quad v_n+mr^{m-2}v_n+mxr^{m-2}\frac{\partial v_n}{\partial x}$$

$$+mxr^{m-2}\frac{\partial v_n}{\partial x}+r^m\frac{\partial^2 v_n}{\partial x^2}$$

$$=m(m-2)x^2r^{m-4}\quad v_n+mr^{m-2}v_n+2mxr^{m-2}\frac{\partial v_n}{\partial x}$$

$$+r^m\frac{\partial^2 v_n}{\partial x^2}\qquad\ldots\qquad(1)$$

Similar expressions for $\dfrac{\partial^2}{\partial y^2}\left(r^m v_n\right)$ and $\dfrac{\partial^2}{\partial z^2}\left(r^m v_n\right)$

$\therefore$ Adding we get

$$\nabla^2(r^m v_n)=m(m-2)(x^2+y^2+z^2)r^{m-4}v_n+3mr^{m-2}v_n$$

$$+2mr^{m-2}\sum\left(x\frac{\partial v_n}{\partial x}\right)+r^m\sum\left(\frac{\partial^2 v_n}{\partial x^2}\right)$$

$$=m(m-2)r^2.\,r^{m-4}.\,v_n+3mr^{m-2}.\,v_n$$

$$+2mr^{m-2}.\,nv_n+r^m\nabla^2 v_n\quad\because\sum\frac{\partial^2 v_n}{\partial x^2}=\nabla^2 v_n$$

$$=m(m+2n+1)r^{m-2}.\,v_n+r^m\nabla^2 v_n.$$

Deduction. If $\nabla^2 v_n=0.$

then $\nabla^2(r^m v_n)=0$

if $m(m+2n+1)r^{m-2}v_n=0.$

i.e., if $m+2n+1=0$

i.e., if $m=-2n-1.$

So for $m=-2n-1$, we must have

$\nabla^2(r^m v_n)=0$

Hence $\nabla^2(r^{-2n-1}.\,v_n)=0.$

Exercise 4

1. If $x=a\cos u\cosh v$, $y=a\sin u\sinh v$, prove that

$$\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{2}a^2(\cosh 2v-\cos 2u).$$

2. If $x=r\cos\theta$, $y=r\sin\theta$ where r and θ are functions of t alone, prove that

$$x\frac{dy}{dt}-y\frac{dx}{dt}=r^2\frac{d\theta}{dt}.$$

3. If V is a function of r alone, where $r^2=x_1^2+x_2^2+\ldots+x_n^2$, prove that

$$\frac{\partial^2 V}{\partial x_1^2}+\frac{\partial^2 V}{\partial x_2^2}+\cdots+\frac{\partial^2 V}{\partial x_n^2}=\frac{d^2 V}{dr^2}+\frac{n-1}{r}\cdot\frac{dV}{dr}$$

4. If $x=r\cos\theta$, $y=r\sin\theta$, prove that

(i) $\dfrac{\partial^2\theta}{\partial x^2}+\dfrac{\partial^2\theta}{\partial y^2}=0$

(ii) $\dfrac{\partial^2 r}{\partial x^2}+\dfrac{\partial^2 r}{\partial y^2}=\dfrac{1}{r}\left\{\left(\dfrac{\partial r}{\partial x}\right)^2+\left(\dfrac{\partial r}{\partial y}\right)^2\right\}$

5. If $V=f(x, y)$ where $x=\xi\cos\alpha-\eta\sin\alpha$ and $y=\xi\sin\alpha+\eta\cos\alpha$, then prove that

$$\frac{\partial^2 V}{\partial x^2}\cdot\frac{\partial^2 V}{\partial y^2}-\frac{\partial^2 V}{\partial x\,\partial y}=\frac{\partial^2 V}{\partial\xi^2}\cdot\frac{\partial^2 V}{\partial\eta^2}-\left(\frac{\partial^2 V}{\partial\xi\,\partial\eta}\right)^2.$$

6. If V be a homogeneous function of x, y, z of degree n and if $u=(x^2+y^2+z^2)^{-\frac{1}{2}(n+1)}$, then prove that

$$\frac{\partial}{\partial x}\left(V\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(V\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(V\frac{\partial u}{\partial z}\right)=0.$$

7. Let $\phi=\phi(x, y)$ where $x=e^v\sec u$, $y=e^v\tan u$, prove that

$$\cos u\left(\frac{\partial^2\phi}{\partial u.\partial v}-\frac{\partial\phi}{\partial u}\right)=xy\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+(x^2+y^2)\frac{\partial^2\phi}{\partial x.\partial y}.$$

8. If $\phi=\phi(x, y)$ where $x=e^u\sec v$, $y=e^u\tan v$, prove that

$$\left(\frac{\partial\phi}{\partial x}\right)^2-\left(\frac{\partial\phi}{\partial y}\right)^2=\frac{1}{e^{2u}}\left\{\left(\frac{\partial\phi}{\partial u}\right)^2-\cos^2 v\left(\frac{\partial\phi}{\partial v}\right)^2\right\}$$

9. A function $f(x, y)$ when expressed in terms of the new variables u, v defined by $x=\frac{1}{2}(u+v)$, $y^2=uv$ becomes $g(u, v)$; prove that

$$\frac{\partial^2 g}{\partial u\,\partial v}=\frac{1}{4}\left\{\frac{\partial^2 f}{\partial x^2}+2\frac{x}{y}\frac{\partial^2 f}{\partial x\,\partial y}+\frac{1}{y}\frac{\partial^2 f}{\partial y^2}\right\}$$ (*C. H. 1965*)

10. Let D stands for the operator $x\dfrac{\partial}{\partial x}+y\dfrac{\partial}{\partial y}$ or, $r\dfrac{\partial}{\partial r}$ in polar, prove

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x.\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=D(D-1)u.$$

If further $x=r\cos\theta$, $y=r\sin\theta$ and $r=e^z$ then deduce that

$$\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial u}{\partial z}=D(D-1)u.$$

CHAPTER V

EXPANSION OF FUNCTIONS

5.1. Mean Value Theorem for a function of two variables. If $f(x, y)$ be a differentiable function of the two variables x and y defined in a certain domain R containing the points (a, b) and $(a+h, b+k)$ so that the line joining these two points lies wholly in R, then

$$f(a+h, b+k)=f(a, b)+hf_x(a+\theta h, b+\theta k)$$
$$+kf_y(a+\theta h, b+\theta k)$$

where $0<\theta<1$.

Proof. Let $x=a+ht$, $y=b+kt$, so that

$$f(x, y)=f(a+ht, b+kt)$$
$$=F(t), \text{ a composite function of a single variable } t.$$

Then applying Mean Value Theorem of a single variable to the function $F(t)$ between 0 and t, we get

$$F(t)-F(0)=tF'(\theta t),\ 0<\theta<1. \quad \cdots \quad (1)$$

But $F'(t)=\dfrac{\partial f}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$

$$=hf_x(a+ht, b+kt)+kf_y(a+ht, b+kt)$$

$\therefore$ from (1)

$$F(t)-F(0)=t\{hf_x(a+h\theta t, b+k\theta t)$$
$$+kf_y(a+h\theta t, b+k\theta t)\}$$

Putting $t=1$

$$F(1)-F(0)=hf_x(a+\theta h, b+\theta k)+kf_y(a+\theta h, b+\theta k)$$

But $F(1)=f(a+h, b+k)$

and $F(0)=f(a, b)$

Hence, we get

$$f(a+h, b+k)=f(a, b)+hf_x(a+\theta h, b+\theta k)$$
$$+kf_y(a+\theta h, b+\theta k)$$

where $0<\theta<1$.

5.2. Taylor's Theorem for a function of two variables.

If $f(x; y)$ possess continuous partial derivatives of nth order in any neighbourhood of a point (a, b) and if $(a+h, b+k)$ be any point of this neighbourhood, then there exists a positive number $\theta<1$ such that

$$f(a+h, b+k)=f(a\ b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a, b)$$

$$+\frac{1}{\lfloor 2}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a, b)+\cdots+\frac{1}{\lfloor n-1}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a, b)$$

$$+\frac{1}{\lfloor n}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k)$$

where $0<\theta<1$.

Lemma. Let $z=f(x, y)$ and $x=a+ht$, $y=b+kt$, so that z is a composite function of a single variable t.

$$\therefore\quad \frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}.$$

$$=\frac{\partial z}{\partial x}.h+\frac{\partial z}{\partial y}.k$$

$$=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)z$$

$$\frac{d^2z}{dt^2}=\frac{d}{dt}\left(\frac{dz}{dt}\right)$$

$$=\frac{d}{dt}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)z$$

$$=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)z$$

$$=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 z.$$

Proceeding in this way, we can write

$$\frac{d^nz}{dt^n}=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n z.$$

Proof of Taylor's Theorem.

We write $f(x, y)=f(a+ht, b+kt)=F(t)$ ... (1)

Now applying Maclaurin's Theorem to the function $F(t)$ of the single variable t,

there exists a positive number $\theta<1$ such that

$$F(t)=F(0)+tF'(0)+\frac{t^2}{\underline{|2}}F''(0)+\cdots+\frac{t^{n-1}}{\underline{|n-1}}F^{n-1}(0)$$

$$+\frac{t^n}{\underline{|n}}F^n(\theta t). \quad \cdots \quad (2)$$

or, for $t=1$, we get

$$F(1)=F(0)+F'(0)+\frac{1}{\underline{|2}}F''(0)+\cdots+\frac{1}{\underline{|n-1}}F^{n-1}(0)$$

$$+\frac{1}{\underline{|n}}F^n(\theta) \quad \cdots \quad (3)$$

Now from (1)

$$F(1)=f(a+h, b+k)$$

$$F(0)=f(a, b).$$

Also from the lemma

$$F^n(t)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x, y)$$

For $t=0$, we have $x=a$, $y=b$

$$\therefore \quad F'(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a, b)$$

$$F''(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$\cdots \quad \cdots \quad \cdots$$

$$F^{n-1}(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a, b)$$

$$F^n(\theta)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k)$$

Putting these values in (3) we get

$$f(a+h, b+k)=f(a, b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a, b)$$

$$+\frac{1}{\lfloor 2}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a, b)+\cdots$$

$$\cdots+\frac{1}{\lfloor n-1}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a, b)$$

$$+\frac{1}{\lfloor n}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k)$$

for $0<\theta<1$.

Cor. 1. Replacing a and b by x and y, the Taylor's Formula may be written in the form

$$f(x+h, y+k)=f(x, y)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x, y)$$

$$+\frac{1}{\lfloor 2}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x, y)+\cdots\cdots+\frac{1}{\lfloor n-1}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(x, y)$$

$$+\frac{1}{\lfloor n}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x+\theta h, y+\theta k) \text{ where } 0<\theta<1.$$

This formula may be extended for any number of variables. Thus for three variables x, y, z we can write

$$f(x+h, y+k, z+l)=f(x, y, z)$$

$$+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}+l\frac{\partial}{\partial z}\right)f(x, y, z)+\frac{1}{\lfloor 2}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}+l\frac{\partial}{\partial z}\right)^2 f(x, y, z)$$

$$+\cdots\cdots+\frac{1}{\lfloor n-1}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}+l\frac{\partial}{\partial z}\right)^{n-1} f(x, y, z)$$

$$+\frac{1}{\lfloor n}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}+l\frac{\partial}{\partial z}\right)^n f(x+\theta h, y+\theta k, z+\theta l) \text{ where } 0<\theta<1.$$

5·3. Maclaurin's Theorem for a function of two variables.

Let $f(x, y)$ be a functions of two independent variables x and y and let

(*i*) (x, y) be a point in the neighbourhood of $(0, 0)$

(*ii*) $f(x, y)$ has derivatives up to $(n-1)$th order and all continuous in $0\leq h\leq x$, $0\leq k\leq y$

(*iii*) $f(x, y)$ is differentiable up to nth order in

$$0<h<x,\ 0<k<y,$$

then there exists a positive number $\theta<1$ such that

$$f(x, y)=f(0, 0)+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0, 0)+\frac{1}{\lfloor 2}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f(0, 0)+$$

$$\cdots\frac{1}{\lfloor n-1}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{n-1} f(0, 0)+\frac{1}{\lfloor n}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^n f_{(\theta x, \theta y)}$$

$$\text{for } 0<\theta<1,$$

where $\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0, 0)$ means after calculating partial derivatives of $f(x, y)$ its value taken by putting $x=0$, $y=0$.

By Taylor's Theorem of two variables, we know that if (*i*) $f(x, y)$ has derivatives upto $(n-1)$th order and continuous in $a\leqslant x\leqslant a+h$, $b\leqslant y\leqslant b+k$

(*ii*) $f(x, y)$ is differentiable up to the nth order in $a<x<a+h$, $b<y<b+k$,

then there exists a positive number $\theta<1$ such that

$$f(a+h, b+k)=f(a, b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a, b)$$

$$+\frac{1}{\lfloor 2}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a, b)+\cdots+\frac{1}{\lfloor n-1}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a, b)$$

$$+\frac{1}{\lfloor n}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k),\ 0<\theta<1.$$

Putting $a=b=0$, and changing h, k by x and y the above Taylor's statement reduces to Maclaurin's statement and the expansion becomes

$$f(x, y)=f(0, 0)+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0, 0)+\frac{1}{\lfloor 2}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0, 0)+$$

$$\cdots+\frac{1}{\lfloor n-1}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{n-1} f(0, 0)+\frac{1}{\lfloor n}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^n f(\theta x, \theta y),$$

$$0<\theta<1.$$

Note. The Maclaurin's expansion can be put in the following convenient form

$$f(x, y)=f(u, v)+\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)f(u,v)+\frac{1}{\underline{|n}}\left\{x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right\}^2 f(u, v)+$$

$$\cdots+\frac{1}{\underline{|n-1}}\left\{x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right\}^{n-1} f(u, v)+\left\{x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right\}^n f(u, v)$$

where after differentiation u, v are to be replaced by (0, 0) except in the last term, where u, v are to be replaced by θx and θy.

Illustrative Examples :

Ex. 1. If the partial derivatives f_x, f_y of a function f(x, y) exist and have the value zero at every point in its domain, prove that f(x, y) is a constant. (C. H. 1961)

By Mean Value Theorem

$$f(x+h, y+k)-f(x, y)=hf_x(x+\theta h, y+\theta h)$$
$$+kf_y(x+\theta h, y+\theta k),\ 0<\theta<1$$
$$=hf_x(\xi, \eta)+kf_y(\xi, \eta) \quad \cdots \quad (1)$$

where (ξ, η) is an intermediate point on the line joining the two points (x, y) and $(x+h, y+k)$ in the domain.

But by the given condition $f_x(\xi, \eta)=0$, $f_y(\xi, \eta)=0$ for every point in the domain.

$\therefore$ from (1) $f(x+h, y+k)-f(x, y)=0$ in the domain

$\therefore$ $f(x, y)$ is constant in the domain.

Ex. 2. Prove that

$$\mathbf{\sin x \sin y = xy - \frac{1}{6}\{(x^3+3xy^2)\cos \theta x \sin \theta y}$$
$$\mathbf{+(y^3+3xy^2)\sin \theta x \cos \theta y\}} \quad \text{for } 0<\theta<1.$$

Let $f(x, y)=\sin x \sin y$, then $f(0, 0)=0$.

$$\therefore \quad \left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)f(u, v)=\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)\sin u \sin v$$

$$=x \sin v \cos u+y \sin u \cos v$$
$$=0 \quad \text{for } u=v=0.$$

$$\frac{1}{\underline{|2}}\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)^2 f(u,v)=\frac{1}{\underline{|2}}\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)^2 \sin u \sin v$$

$$=\frac{1}{2}\left\{x^2\frac{\partial^2}{\partial u^2}(\sin u \sin v)+2xy\frac{\partial^2}{\partial u\,\partial v}(\sin u \sin v)+y^2\frac{\partial^2}{\partial v^2}(\sin u \sin v)\right\}$$

$$=\frac{1}{2}\{-x^2 \sin u \sin v+2xy \cos u \cos v-y^2 \sin u \sin v\}$$

$$=xy \text{ for } u=v=0$$

$$\frac{1}{\underline{|3}}\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)^3 f(u,v)$$

$$=\frac{1}{6}\left\{x^3\frac{\partial^3}{\partial u^3}(\sin u \sin v)+3x^2y\frac{\partial^3}{\partial u^2 . \partial v}(\sin u \sin v)+3xy^2\frac{\partial^3}{\partial u\,\partial v^2}(\sin u \sin v)+y^3\frac{\partial^3}{\partial v^3}(\sin u \sin v)\right\}$$

$$=\frac{1}{6}\{-x^3 \cos u \sin v-3x^2y \sin u \cos v-3xy^2 \cos u \sin v-y^3 \sin u \cos v\}$$

$$=-\frac{1}{6}\{(x^3+3xy^2)\cos \theta x \sin \theta y+(y^3+3x^2y)\sin \theta x \cos \theta y\} \text{ for } u=\theta x,\ v=\theta y.$$

Now putting in the Maclaurin's Theorem, we get

$$\sin x \sin y=xy-\frac{1}{6}\{(x^3+3xy^2)\cos \theta x \sin \theta y+(y^3+3x^2y)\sin \theta x \cos \theta y\}$$

where $0<\theta<1$

Ex. 3. Show that

$$\mathbf{e^{ax} \sin by=by+abxy+e^{x\theta a}\{(a^3x^3-3ab^2xy^2)\sin b\theta y+(3a^2bx^2y-b^3y^3)\cos b\theta y\},\ 0<\theta<1}$$ *(C. H. 1964)*

Let $f(x, y)=e^{ax} \sin by$, then $f(u, v)=e^{au} \sin bv$

$\therefore\ f(0, 0)=0.$

$$\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right) f(u, v)=\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)e^{au} \sin bv$$

$$=xae^{au} \sin bv+ybe^{au} \cos bv$$

$$=by \text{ for } u=v=0.$$

$$\frac{1}{\lfloor 2}\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)^2 e^{au} \sin bv$$

$$=\frac{1}{2}\left\{x^2\frac{\partial^2}{\partial u^2}e^{au} \sin bv+2xy\frac{\partial^2}{\partial u\partial v}e^{au} \sin bv + y^2\frac{\partial^2}{\partial v^2}e^{au} \sin bv\right\}$$

$$=\tfrac{1}{2}(x^2a^2e^{au} \sin bv+2xyabe^{au} \cos bv-y^2b^2e^{au} \sin bv)$$

$$=abxy \quad \text{for } u=v=0.$$

$$\frac{1}{\lfloor 3}\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)^3 e^{au} \sin bv$$

$$=\frac{1}{6}\left\{x^3\frac{\partial^3}{\partial u^3}(e^{au} \sin bv)+3x^2y\frac{\partial^3}{\partial u^2\partial v}(e^{au} \sin bv) +3xy^2\frac{\partial^3}{\partial u\partial v^2}(e^{au} \sin bv)+y^3\frac{\partial^3}{\partial v^3}(e^{au} \sin bv)\right\}$$

$$=\frac{1}{6}\{x^3a^3e^{au} \sin bv+3x^2ya^2be^{au} \cos bv -3xy^2ab^2e^{au} \sin bv-y^3b^3e^{au} \cos bv\}$$

$$=\frac{1}{6}e^{au}\{(a^3x^3-3ab^2xy^2) \sin bv+(3a^2bx^2y-b^3y^3) \cos bv\}$$

$$=\frac{1}{6}e^{a\theta x}\{(a^3x^3-3ab^2xy^2) \sin b\theta y+(3a^2bx^2y-b^3y^3) \cos b\theta y\}.$$

putting $u=\theta x$ and $v=\theta y$

Now putting in the Maclaurin's Theorem we get the result.

Exercise 5

1. If the partial derivatives f_x, f_y of the function $f(x, y)$ are continuous in R, prove that

$$f(x+h, y+k)-f(x, y)=hf_x(\xi, \eta)+kf_y(\xi, \eta)$$

where (ξ, η) is a point in the line joining (x, y) and $(x+h, y+k)$.

(C. H. 1961)

2. Prove that for $0<\theta<1$

$$f(x+h,\ y+k)=f(x,\ y)+\frac{1}{|\underline{2}}\left[\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right]f(x,\ y)$$
$$+\frac{1}{|\underline{\ }}\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right](x+\theta h,\ y+\theta k),$$

Also state the conditions under which the above expansion is valid.

3. If $f(x,\ y)=f(a+ht,\ b+kt)=F(t)$ where $a,\ b,\ h,\ k$ are constants, show that for $0<\theta<1$

$$F^{n}(\theta)=\left[h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right]^{n}f(a+\theta h,\ b+\theta k).$$

4. If $z=f(x,\ y)$ and $x=a+ht,\ y=b+kt$ where $a,\ b,\ h,\ k$ are constants, show that

$$\frac{d^{n}z}{dt^{n}}=\left\{h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right\}^{n}_{z}$$
$$=h^{n}\frac{\partial^{n}z}{\partial x^{n}}+\binom{n}{1}h^{n-1}k\frac{\partial^{n}z}{\partial x^{n-1}\partial y}+\ldots$$
$$\ldots+\binom{n}{r}h^{n-r}k^{r}\frac{\partial^{n}z}{\partial x^{n-r}\partial y^{r}}+\ldots+k^{n}\frac{\partial^{n}z}{\partial y^{n}}.$$

5. Show that for $0<\theta<1$

$$e^{ax}\cos by=1+ax+\frac{1}{|\underline{2}}(a^{2}x^{2}-b^{2}y^{2})$$
$$+\frac{e^{a\theta x}}{|\underline{3}}\{(a^{3}x^{3}-3ab^{2}xy^{2})\cos b\theta y-(3a^{2}bx^{2}y-b^{3}y^{3})\sin b\theta y\}$$

CHAPTER VI

MAXIMA AND MINIMA

6.1. Extreme value of a function of two variables.

If $f(a, b)$ be a maximum or minimum value of $f(x, y)$ at (a, b), then $f(a, b)$ is said to be an extreme value of the function $f(x, y)$.

If there exist some neighbourhood of (a, b) such that for every point $(a+h, b+k)$ of this neighbourhood

$$f(a, b) > f(a+h, b+k)$$

then $f(a, b)$ is said to be a maximum value of $f(x, y)$ at (a, b).

If there exist some neighbourhood of (a, b) such that for every point $(a+h, b+k)$ of this neighbourhood

$$f(a, b) < f(a+h, b+k)$$

then $f(a, b)$ is said to be minimum value of $f(x, y)$ at (a, b).

6.2. The necessary conditions for $f(a, b)$ to be an extreme value of $f(x, y)$ are that

$$f_x(a, b)=0, \quad f_y(a, b)=0$$

Suppose $f(a, b)$ is an extreme value of the function $f(x, y)$ at (a, b).

Then $f(a, b)$ is also an extreme value of the function $f(x, b)$ at $x=a$

But $f(x, b)$ being a function of a single variable x, the necessary condition for its extreme value is $f_x(x, b)=0$ at $x=a$

$\therefore \; f_x(a, b)=0.$

Similarly, $f_y(a, b)=0.$

Note : $f(a, b)$ is said to be a stationary value of $f(x, y)$ at (a, b) if at this point

$$f_x(a, b)=0, \quad f_y(a, b)=0$$

But these are the conditions of an extreme value. Thus every extreme value is a stationary value but the converse is not always true.

6.3. Sufficient condition for the extremum of a function f(x, y) at (a, b).

If $f_x(a, b)=0, \quad f_y(a, b)=0$

and $f_{x^2}(a, b)=A, \quad f_{xy}(a, b)=B, \quad f_{y^2}(a, b)=C$

then (1) $f(a, b)$ is a maximum value of $f(x, y)$ at (a, b) if $AC-B^2>0$ and $A<0$.

(2) $f(a, b)$ is a minimum value of $f(x, y)$ at (a, b) if $AC-B^2>0$ and $A>0$.

(3) $f(a, b)$ is neither a maximum nor a minimum value of $f(x, y)$ at (a, b) if $AC-B^2<0$

(4) the case is doubtful and needs further investigation if $AC-B^2=0$.

Proof. By Taylor's Theorem with remainder after three terms we have

$$f(a+h, b+k)=f(a, b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a, b)$$

$$+\frac{1}{\underline{|2}}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$+\frac{1}{\underline{|3}}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(a+\theta h, b+\theta k)$$

$$=f(a, b)+hf_x(a, b)+kf_y(a, b)$$

$$+\tfrac{1}{2}\{h^2f_{x^2}(a, b)+2hkf_{xy}(a, b)+k^2f_{y^2}(a, b)\}$$

$$+\tfrac{1}{6}\{h^3f_{x^3}(u, v)+3h^2kf_{x^2y}(u, v)+3hk^2f_{xy^2}(u, v)+k^3f_{y^3}(u, v)\}$$

where $u=a+\theta h, \quad v=b+\theta k$.

or, $$f(a+h, b+k)-f(a, b)=hf_x(a, b)+kf_y(a, b)$$

$$+\tfrac{1}{2}\{h^2f_{x^2}(a, b)+2hkf_{xy}(a, b)+k^2f_{y^2}(a, b)\}+R_n$$

where R_n contains the terms of third degree in h and k which are very small.

Now by the given conditions, since $f_x(a, b)=0$ and $f_y(a, b)=0$, we get

$$f(a+h, b+k)-f(a, b)=\tfrac{1}{2}(Ah^2+2Bhk+Ck^2)+R_n \quad \cdots \quad (1)$$

Now for sufficiently small values of h and k the value of R_n can be made so small, that the sign of the $R.H.S.$ of (1) will be the same as that of

$$Ah^2+2Bhk+Ck^2.$$

So neglecting R_n in (1) we write

$$f(a+h, b+k)-f(a, b)=\frac{1}{2}(Ah^2+2Bhk+Ck^2)$$

$$=\frac{1}{2A}(A^2h^2+2ABhk+ACk^2)$$

$$=\frac{1}{2A}\{(Ah+Bk)^2+(AC-B^2)k^2\} \quad \cdots \quad (2)$$

Case I. Let $AC-B^2>0$, then neither A nor C is zero. So from (2) we get

$$f(a+h, b+k)-f(a, b)=\frac{1}{2A}\times a \text{ non zero positive quantity} \quad \cdots \quad \cdots \quad \cdots \quad (3)$$

$[\because \quad h\neq 0, k\neq 0.]$

So if besides $AC-B^2>0$, we have $A<0$ then from (3)

$$f(a+h, b+k)-f(a, b)<0$$

i.e., $f(a, b)>f(a+h, b+k)$.

Hence $f(a, b)$ is a maximum value.

Again if besides $AC-B^2>0$ we have $A>0$ then from (3)

$$f(a+h, b+k)-f(a, b)>0$$

i.e., $f(a, b)<f(a+h, b+k)$

Hence, $f(a, b)$ is a minimum value.

Case II. Let $AC-B^2<0$ and suppose $A\neq 0$, then $(AC-B^2)k^2<0$.

Let $(AC-B^2)k^2=-\alpha$ where α is a positive quantity.

Then from (3)

$$f(a+h,\ b+k)-f(a,\ b)=\frac{1}{2A}\{(Ah+Bk)^2-\alpha\}$$

$$=\frac{1}{2A}\times a \text{ positive or a negative}$$

quantity.

So for any sign of A, the $R.H.S.$ may be positive or negative.

i.e., $f(a+h,\ b+k)-f(a,\ b)>0$ or <0 for the same sign of A.

Hence, $f(a,\ b)$ cannot be an extreme value.

Similar is the proof if $C\neq 0$.

Again if both $A=0$ and $C=0$ then from (1)

neglecting R_n, $f(a+h,\ b+k)-f(a,\ b)=\frac{1}{2}\ 2Bhk=Bhk.$

$\therefore$ $f(a+h,\ b+k)-f(a,\ b)$ assumes values with different signs and hence $f(a,\ b)$ is not an extreme value.

Case III. Let $AC-B^2=0$ and suppose $A\neq 0$

Then from (1)

$$f(a+h,\ b+k)-f(a,\ b)=\frac{1}{2}(Ah^2+2Bhk+Ck^2)+R_n$$

$$=\frac{1}{2A}\{(Ah+Bk)^2+(AC-B^2)k^2\}+R_n$$

$$=\frac{1}{2A}(Ah+Bk)^2+R_n$$

$$=R_n \text{ when } Ah+Bk=0.$$

So the nature of the sign of $f(a+h,\ b+k)-f(a,\ b)$ depends on R_n. The case is therefore doubtful.

If $A=0$ then from the condition $AC-B^2=0$ we get $B=0$

$$\therefore\quad f(a+h,\ b+k)-f(a,\ b)=\tfrac{1}{2}Ck^2+R_n$$

$$=R_n \text{ if } k=0 \text{ whether } h \text{ is zero or not.}$$

This case therefore is also doubtful.

Illustrative Examples :

Ex. 1. Find the minimum value of

$$\mathbf{x^2+y^2+(x+y+1)^2}$$ **(C. H. 1966)**

Let $f(x, y)=x^2+y^2+(x+y+1)^2$

Then $f_x=2x+2(x+y+1)=4x+2y+2$

$f_y=2y+2(x+y+1)=2x+4y+2.$

For $f_x=0$ and $f_y=0$ we have

$4x+2y+2=0$

and $2x+4y+2=0$

Solving these two equations $x=-\frac{1}{3}$, $y=-\frac{1}{3}$.

So $f(x, y)$ may have an extremum at $x=-\frac{1}{3}$, $y=-\frac{1}{3}$.

Now $A=f_{xx}=4,$

$B=f_{xy}=2$

$C=f_{yy}=4$

$\because$ $AC-B^2=16-4=12>0$

and $A>0$, the function $f(x,y)$ is minimum at $(-\frac{1}{3}, -\frac{1}{3})$.

The minimum value of $f(x, y)$

$=(-\frac{1}{3})^2+(-\frac{1}{3})^2+(-\frac{1}{3}-\frac{1}{3}+1)^2$

$=\frac{1}{3}.$

Ex. 2. Find the maximum or minimum value of $\mathbf{f(x, y)=x^2+xy+y^2+ax+by}$ and determine whether the value you get is maximum or minimum. (C. H. 1964)

Here $f_x=2x+y+a$, $f_y=x+2y+b.$

For $f_x=0$ and $f_y=0$ we get

$2x+y+a=0$

$x+2y+b=0$

$\therefore$ Solving $x=\frac{1}{3}(b-2a)$, $y=\frac{1}{3}(a-2b)$

Now $A=f_{xx}=2,$

$B=f_{xy}=1$

And $C=f_{yy}=2$

Since, $AC-B^2=4-1=3>0$ and $A>0$,

$f(x, y)$ is minimum at $x=\frac{1}{3}(b-2a)$, $y=\frac{1}{3}(a-2b)$.

The minimum value of $f(x, y)$

$$=\tfrac{1}{9}(b-2a)^2+\tfrac{1}{9}(b-2a)(a-2b)+\tfrac{1}{9}(a-2b)^2$$
$$+\frac{a}{3}(b-2a)+\frac{b}{3}(a-2b)$$
$$=\tfrac{1}{9}(-3b^2-3a^2+3ab)$$
$$=\tfrac{1}{3}(ab-a^2-b^2).$$

Ex. 3. Find all the maxima and minima of the function $\mathbf{x^3+y^3-63(x+y)+12xy}$. [*C. H 1968*]

Let $f(x, y)=x^3+y^3-63x-63y+12xy$.

Then $f_x=3x^2-63+12y$

$f_y=3y^2-63+12x$.

$\therefore$ $f_x=0$, $f_y=0$ give

$$x^2+4y-21=0$$
$$y^2+4x-21=0.$$

Substrating 2nd from the 1st

$$(x^2-y^2)+4(y-x)=0$$

or, $(x-y)(x+y-4)=0$

$\therefore$ $x-y=0$ or $x+y-4=0$.

So we get two set of equations

$$x^2+4y-21=0 \text{ and } x^2+4y-21=0$$
$$x-y=0 \qquad\qquad x+y-4=0.$$

Solving these two set of equations, we get the following sets of points as root of $f_x=0$ and $f_y=0$.

$$(3, 3), (-7, -7), (5, -1) \text{ and } (-1, 5)$$

Now $A=f_{xx}=6x$; $B=12$; $C=6y$.

$\therefore$ At (3, 3) we have $A=18$, $B=12$ and $C=18$

So that $AC-B^2=18^2-12^2>0$ and A is positive.

$\therefore$ $f(x, y)$ is minimum at (3, 3).

At $(-7, -7)$ we have $A=-42$, $B=12$, $C=-42$

So that $AC-B^2=42^2-144>0$ and A is negative

$\therefore$ $f(x,y)$ is maximum at $(-7, -7)$.

At $(5, 1)$ we have $A=30$, $B=12$ and $C=-6$

So that $AC-B^2=-180-144=$ negative

$\therefore$ $f(x, y)$ has no extremum at $(5,-1)$.

At $(-1, 5)$, $A=-6$, $B=12$ and $C=30$

So that $AC-B^2=-180-144=$ negative

$\therefore$ $f(x, y)$ has no extremum at $(-1, 5)$.

Ex. 4. Show that $u=x^2y^4(1-x-y)^6$ defined in the region $x\geqslant 0$, $y\geqslant 0$, $x+y\leqslant 1$ attain its greatest at an interior point of the region and find this greatest value.

$$u_x=2xy^4(1-x-y)^6-6x^2y^4(1-x-y)^5$$
$$=2xy^4(1-x-y)^5(1-4x-4y)$$
$$u_y=4x^2y^3(1-x-y)^6-6x^2y^4(1-x-y)^5$$
$$=2x^2y^3(1-x-y)^5(2-2x-5y).$$

$\therefore$ $u_x=0$ and $u_y=0$ give

$$x=0,\ y=0,\ x+y=1,\ 4x+4y=1,\ 2x+5y=2$$

Solving these equations we get the following set of points as solutions of $f_x=0$ and $f_y=0$, $(0, 0)$, $(0, 1)$, $(0, \frac{2}{5})$, $(1, 0)$, $(\frac{1}{4}, 0)$, $(\frac{1}{6}, \frac{1}{3})$, of which $(\frac{1}{6}, \frac{1}{3})$ is an interier point of the given region. So we are to examine the point $(\frac{1}{6}, \frac{1}{3})$ for extremum.

Now $A=u_{xx}=2y^4(1-x-y)^5(1-4x-y)$
$$-10xy^4(1-x-y)^4(1-4x-y)-8xy^4(1-x-y)^5$$
$$=-\frac{1}{54\times 36} \text{ at } x=\frac{1}{6},\ y=\frac{1}{3}.$$

$$B=u_{xy}=8xy^3(1-x-y)^5(1-4x-y)$$
$$-10xy^4(1-x-y)^4(1-4x-y)-2xy^4(1-x-y)^5$$
$$=-\frac{1}{54\times 144} \text{ at } \left(\frac{1}{6}, \frac{1}{3}\right).$$

$$C = u_{yy} = 6x^2y^2(1-x-y)^5(2-2x-5y)$$
$$-10x^2y^3(1-x-y)^4(2-2x-5y)$$
$$-10x^2y^3(1-x-y)^5$$
$$= -\frac{10}{36\times 27}\left(1-\frac{1}{6}-\frac{1}{3}\right)^5$$
$$= -\frac{5}{18\times 27}\left(\frac{3}{6}\right)^5 \text{ at } \left(\frac{1}{6}, \frac{1}{3}\right)$$

$$\therefore \quad AC - B^2 = \left(-\frac{1}{54\times 36}\right)\left(-\frac{5}{18\times 27}\frac{1}{2^5}\right) - \left(\frac{2}{54\times 144}\right)^2$$

= positive i.e., >0 and A is negative.

$\therefore \quad f(x, y)$ is maximum at $\left(\frac{1}{6}, \frac{1}{3}\right)$.

So that maximum $u = \frac{1}{36}\cdot\frac{1}{81}\left(1-\frac{1}{6}-\frac{1}{3}\right)^6 = \frac{1}{36\times 81\times 64}$.

Ex. 5. Investigate the maximum and minimum of

$$U = 2(x-y)^2 - x^4 - y^4$$

leaving aside any doubtful case that may arise.

$$U_x = 4(x-y) - 4x^3$$
$$U_y = -4(x-y) - 4y^3.$$

$\therefore \quad U_x = 0$ and $U_y = 0$ give

$(x-y) - x^3 = 0 \quad \cdots \quad (1)$ and $x - y + y^3 = 0 \quad \cdots \quad (2)$

Substracting 1st equation from 2nd

$y^3 + x^3 = 0 \qquad \therefore \quad y = -x.$

putting $y = -x$ in (1) we get $2x - x^3 = 0$

$\therefore \quad x = 0$ or, $\pm\sqrt{2}$ and $y = 0$ or, $\mp\sqrt{2}$.

$\therefore$ The points are $(0, 0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$.

Now $\quad A = U_{xx} = 4 - 12x^2$

$B = U_{xy} = -4$

$C = 4 - 12y^2$

$\therefore$ At $(0, 0)$ we have $A = 4$, $B = -4$, $C = 4$

So that $\quad AC - B^2 = 16 - 16 = 0$, a doubtful case.

At $(\sqrt{2}, -\sqrt{2})$ we have $A=4-24=-20$, $B=-4$ $C=4-24=-20$, so that $AC-B^2=400-16=$positive and A is negative

$\therefore \quad f(x, y)$ is maximum at $(\sqrt{2}, -\sqrt{2})$

Similarly, $f(x, y)$ is maximum at $(-\sqrt{2}, \sqrt{2})$.

6.4. Constrained Maxima and Minima.

1. To find the maximum or minimum value of f(x, y) subject to the additional condition ϕ(x, y)=0. *(C. H. 1964)*

Let $u=f(x, y)$ and suppose u is a differential function in a given domain.

Now, we have the constrained equation

$\phi(x, y)=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$

If $\phi_y \neq 0$, then we can solve (1) for y.

Suppose, solving (1) we get $y=\psi(x)$.

Then the original function is reduced to $u=f(x, \psi)$ a function of x alone.

Differentiating this w. r. to x

$$\frac{du}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial \psi}\frac{d\psi}{dx}.$$

Now for an extremum $\frac{du}{dx}=0$

$$\therefore \quad \frac{\partial f}{\partial x}+\frac{\partial f}{\partial \psi}\frac{d\psi}{dx}=0$$

$$\text{or,} \quad \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0 \quad \ldots \quad \ldots \quad (2)$$

Again from $\phi(x, y)=0$

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \quad \ldots \quad \ldots \quad (3)$$

Eliminating $\frac{dy}{dx}$ from (2) and (3) we get

$$\frac{f_x}{\phi_x}=\frac{f_y}{\phi_y} \quad \text{or,} \quad f_x\phi_y-\phi_x f_y=0$$

$$\text{or,}\quad \frac{\partial(f, \phi)}{\partial(x, y)}=0 \qquad \cdots \qquad \cdots \qquad (4)$$

Equations (1) and (4) will now give the systems of values of x, y for which $f(x, y)$ may be a maximum or a minimum.

2. To find the maximum or minimum value of f(x, y, z) subject to the additional condition ϕ(x, y, z)=0.

We have the constrained equation

$$\phi(x, y, z)=0 \qquad \cdots \qquad \cdots \qquad (1)$$

From (1) solving for z, suppose, we get

$$z=\psi(x, y)$$

The problem is then reduced to find the extreme value of $f(x, y, \psi)$.

We now write $F(x, y)=f(x, y, \psi)$ $\cdots$ (2)

The necessary condition for the extreme of (2) is that F_x and F_y must vanish separately.

$$\therefore\quad \frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial \psi}{\partial x}=0 \qquad \cdots \qquad \cdots \qquad (3)$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial \psi}{\partial y}=0 \qquad \cdots \qquad \cdots \qquad (4)$$

Also from (1) we get

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial z}\cdot\frac{\partial \psi}{\partial x}=0 \qquad \cdots \qquad \cdots \qquad (5)$$

$$\frac{\partial \phi}{\partial y}+\frac{\partial \phi}{\partial z}\cdot\frac{\partial \psi}{\partial y}=0 \qquad \cdots \qquad \cdots \qquad (6)$$

Eliminating $\dfrac{\partial \psi}{\partial x}$ from (3) and (5) we get

$$\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial \phi}{\partial x}}=\frac{\dfrac{\partial f}{\partial z}}{\dfrac{\partial \phi}{\partial z}} \qquad \text{or,}\quad f_x\phi_z-f_z\phi_x=0$$

$$\text{or,}\quad \frac{\partial(f, \phi)}{\partial(x, z)}=0 \qquad \cdots \qquad \cdots \qquad (7)$$

Similarly, eliminating $\frac{\partial \psi}{\partial y}$ from (4) and (6) we get

$$\frac{\partial(f, \phi)}{\partial(y, z)}=0 \qquad \cdots \qquad \cdots \qquad (8)$$

Equations (1), (7) and (8) will now give a system of values of x, y, z for which the given function is extremum.

6.5. Lagranges method of finding stationary values of a function of several variables connected by one or more relations. [*C. H.* *1948*]

Let $u=f(x_1, x_2, \cdots x_n)$ be a function of n variables x_1, $x_2, \cdots x_n$, connected by the m number of constrained equations

$$\left.\begin{array}{l} f_1(x_1, x_2, \cdots x_n)=0 \\ f_2(x_1, x_2, \cdots x_n)=0 \\ \cdots \quad \cdots \quad \cdots \\ f_m(x_1, x_2, \cdots x_n)=0. \end{array}\right\} \cdots \qquad \cdots \qquad (1)$$

For an extremum $du=0$, so from

$u=f(x_1\ x_2 \cdots x_n)$ we get

$$\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2 \cdots +\frac{\partial f}{\partial x_n}dx_n=0 \qquad \cdots \qquad (2)$$

Also from (1)

$$\left.\begin{array}{l} \frac{\partial f_1}{\partial x_1}dx_1+\frac{\partial f_1}{\partial x_2}dx_2+\cdots+\frac{\partial f_1}{\partial x_n}dx_n=0 \\ \frac{\partial f_2}{\partial x_1}dx_1+\frac{\partial f_2}{\partial x_2}dx_2+\cdots+\frac{\partial f_2}{\partial x_n}dx_n=0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ \frac{\partial f_m}{\partial x_1}dx_1+\frac{\partial f_m}{\partial x_2}dx_2+\cdots+\frac{\partial f_m}{\partial x_n}dx_n=0 \end{array}\right\} \cdots \qquad (3)$$

Multiplying (2) by (1) and (3) by $\lambda_1, \lambda_2, \cdots \lambda_m$ respectively and adding we get

$$P_1dx_1+P_2dx_2+\cdots+P_rdx_r+\cdots+P_ndx_n=0 \quad \cdots \quad (4)$$

where $P_r = \frac{\partial f}{\partial x_r} + \lambda_1 \frac{\partial f_1}{\partial x_r} + \lambda_2 \frac{\partial f_2}{\partial x_r} + \cdots + \lambda_m \frac{\partial f_m}{\partial x_r}$

Since $\lambda_1, \lambda_2, \cdots \lambda_m$ are arbitrary we can choose them in such a way that

$$P_1 = 0,\ P_2 = 0 \cdots P_m = 0 \qquad \cdots \qquad \cdots \qquad (5)$$

The equation (4) then becomes

$$P_{m+1} dx_{m+1} + P_{m+2} dx_{m+2} + \cdots + P_n dx_n = 0 \qquad \cdots \qquad (6)$$

Now by virtue of (2) and (3) there remains only $n-m$ variables as independent. So the quantities $dx_{m+1}, dx_{m+2}, \cdots dx_n$ in (6) are all independent and their co-efficient should vanish separately.

$\therefore$ Form (6) we

$$P_{m+1} = 0,\ P_{m+2} = 0 \cdots P_n = 0 \qquad \cdots \qquad \cdots \qquad (7)$$

Thus we get the relations (1), (5) and (7) giving $m+m+(n-m)$ i. e., $m+n$ relations viz. $f_1 = 0, f_2 = 0 \cdots f_m = 0$
$P_1 = 0, P_2 = 0 \cdots P_n = 0$

From these relations we get the m multipliers $\lambda_1, \lambda_2, \cdots \lambda_m$ and the values of $x_1, x_2, \cdots x_n$ for which the given function is extremum.

6.6. If f(x, y, z, w) has an extreme value at (ξ, η, ζ, ρ) subject to the subsidiary conditions

$$\phi(x, y, z, w) = 0,\ \psi(x, y, z, w) = 0$$

and if at that point $\frac{\partial(\phi, \psi)}{\partial(z, w)} \neq 0$

then two numbers λ and μ exist such that at the point (ξ, η, ζ, ρ) the equation

$$f_x + \lambda\phi_x + \mu\psi_x = 0$$
$$f_y + \lambda\phi_y + \mu\psi_y = 0$$
$$f_z + \lambda\phi_z + \mu\psi_z = 0$$
$$f_w + \lambda\phi_w + \mu\psi_w = 0$$

and also the subsidiary conditions are satisfied.

Let $u = f(x, y, z, w) \qquad \cdots \qquad \cdots \qquad (1)$

For an extremum, $du = 0$

$$\therefore \quad \frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz+\frac{\partial f}{\partial w}dw=0 \quad \cdots \quad (2)$$

Also from the subsidiary conditions,

$\phi(x, y, z, w)=0$ and $\psi(x, y, z, w)=0$

We have

$$\frac{\partial \phi}{\partial x}dx+\frac{\partial \phi}{\partial y}dy+\frac{\partial \phi}{\partial z}dz+\frac{\partial \phi}{\partial w}\partial w=0 \quad \cdots \quad (3)$$

And $$\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy+\frac{\partial \psi}{\partial z}\partial z+\frac{\partial \psi}{\partial w}dw=0 \quad \cdots \quad (4)$$

Multiplying (2) by (1), (3) by λ and (4) by μ and adding we get

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}+\mu\frac{\partial \psi}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}+\mu\frac{\partial \psi}{\partial y}\right)dy$$

$$\left(\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}+\mu\frac{\partial \psi}{\partial z}\right)dz+\left(\frac{\partial f}{\partial w}+\lambda\frac{\partial \phi}{\partial w}+\mu\frac{\partial \psi}{\partial w}\right)dw=0 \quad \cdots \quad (5)$$

Since λ and μ are arbitrary, we can choose them in such a way that

$$\left.\begin{aligned} &\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}+\mu\frac{\partial \psi}{\partial z}=0 \\ \text{And} \quad &\frac{\partial f}{\partial w}+\lambda\frac{\partial \phi}{\partial w}+\mu\frac{\partial \psi}{\partial w}=0 \end{aligned}\right\} \quad \cdots \quad \cdots \quad (6)$$

Solving the pair of equations in (6) we get

$$\frac{\lambda}{\psi_z f_w-\psi_w f_z}=\frac{\mu}{\phi_w f_z-\phi_z f_w}=\frac{1}{\phi_z\psi_w-\psi_z\phi_w}$$

$$\therefore \quad \lambda=\frac{\psi_z f_w-\psi_w f_z}{\phi_z\psi_w-\psi_z\phi_w}, \quad \mu=\frac{\phi_w f_z-\phi_z f_w}{\phi_z\psi_w-\psi_z\phi_w}$$

This shows that the above choice of λ, μ is always possible provided

$$\phi_z\psi_w-\psi_z\phi_w\neq 0$$

i. e., $$\frac{\partial(\phi, \psi)}{\partial(z, w)}\neq 0$$

Now from (5) using (6) we get

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}+\mu\frac{\partial \psi}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}+\frac{\partial \phi}{\partial y}\right)dy=0. \quad (7)$$

Since, the four variables x, y, z, w are connected by two equations

$$\phi(x, y, z, w) = 0 \text{ and } \psi(x, y, z, w) = 0$$

it is always possible to find any two of them in terms of the remaining two. Then u can be regarded as the function of two independent variables x and y which can be made to vary independently of each other. Hence for an extremum the co-efficient of dx and dy in (7) must vanish separately. Thus we get another two relations.

$$\left.\begin{aligned} &\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}+\mu\frac{\partial \psi}{\partial x}=0 \\ \text{And } &\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}+\mu\frac{\partial \psi}{\partial y}=0. \end{aligned}\right\} \quad \cdots \quad \cdots \quad (8)$$

Hence for an extremum, equations (1), (6) and (8) determine x, y, z, w, λ, μ.

Hence the theorem.

Ex. 1. Find the maximum or minimum value of $x^m \cdot y^n \cdot z^p$. subject to the condition $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$.

Let $u=x^m y^n z^p$ and $f=\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-1$.

Then by logarithimic differentiation of u

$$\frac{1}{u}du=\frac{m}{x}dx+\frac{n}{y}dy+\frac{p}{z}dz=0$$

Also differentiating f

$$df=-\frac{a}{x^2}dx-\frac{b}{y^2}dy-\frac{c}{z^2}dz=0$$

Thus, we get two equations

$$\frac{m}{x}dx+\frac{n}{y}dy+\frac{p}{z}dz=0 \quad \cdots \quad \cdots \quad (1)$$

$$\text{and } \frac{a}{x^2}dx+\frac{b}{y^2}dy+\frac{c}{z^2}dz=0. \quad \cdots \quad (2)$$

Multiplying (1) and (2) by 1 and λ respectively and adding and then equating co-efficients of dx, dy and dz

$$\frac{m}{x}+\frac{\lambda a}{x^2}=0 \quad \frac{n}{y}+\frac{\lambda b}{y^2}=0, \frac{p}{z}+\frac{\lambda c}{z^2}=0.$$

or, $m+\frac{\lambda a}{x}=0. \quad n+\frac{\lambda b}{y}=0, \quad p+\frac{\lambda c}{z}=0 \quad \cdots \quad (3)$

$\therefore$ adding $(m+n+p)+\lambda\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)=0.$

or, $(m+n+p)+\lambda=0$

$\therefore \quad \lambda=-(m+n+p).$

Hence from (3) $\frac{mx}{a}=\frac{ny}{b}=\frac{pz}{c}=(m+n+p)$

$\therefore \quad x=\frac{a(m+n+p)}{m}, \quad y=\frac{b(m+n+p)}{n},$

$z=\frac{c(m+n+p)}{p}. \quad \cdots \quad \cdots \quad (4)$

$\therefore$ Maximum or minimum value of u

$=\frac{a^m}{m^m}\cdot\frac{b^n}{n^n}\cdot\frac{c^p}{p^p}\cdot(m+n+p)^{m+n+p} \quad \cdots \quad 5)$

Discrimination : Differentiating $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$, taking z as a function of (x, y) we get

$$-\frac{a}{x^2}-\frac{c}{z^2}\frac{\partial z}{\partial x}=0 \qquad \therefore \quad \frac{\partial z}{\partial x}=-\frac{a}{c}\frac{z}{x}$$

Now from $u=x^m y^n z^p$

$\log u=m\log x+n\log y+p\log z$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x}=\frac{m}{x}+\frac{p}{z}\frac{\partial z}{\partial x}$$

$$=\frac{m}{x}-\frac{p}{z}\cdot\frac{az^2}{cx^2}$$

$$=\frac{m}{x}-\frac{paz}{cx^2}.$$

Differentiating again w. r. to x

$$-\frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{u}\frac{\partial^2 u}{\partial x^2}=-\frac{m}{x^2}+\frac{2paz}{cx^3}-\frac{pa}{cx^2}\frac{\partial z}{\partial x}$$

For maximum or minimum $\frac{\partial u}{\partial x}=0$

So, we get

$$\frac{1}{u}\frac{\partial^2 u}{\partial x^2}=-\frac{m}{x^2}+\frac{2paz}{cx^3}-\frac{pa}{cx^2}\left(-\frac{az^2}{cx^2}\right)$$

$$=-\frac{m}{x^2}+\frac{2pa}{cx^2}\left(\frac{mc}{pa}\right)+\frac{pa^2}{c^2x^2}\left(\frac{mc}{pa}\right)^2 \quad \because \quad \frac{z}{x}=\frac{mc}{pa}$$

$$=-\frac{m}{x^2}+\frac{2m}{x^2}+\frac{am^2}{x^2p}$$

$$=\frac{1}{x^2}\left(m+\frac{am^2}{p}\right)$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}>0 \quad \text{for} \quad x=\frac{a(m+n+v)}{m}$$

Hence u is minimum and minimum value of

$$u=\frac{a^m b^n c^p}{m^m \cdot n^n \cdot p^p}(m+n+p)^{m+n+p}.$$

Ex. 2. If $u=x^2+y^2$ and if $ax^2+2hxy+by^2=1$, find the extreme value of u. [*C. H. 1967*]

Here $\quad du=2xdx+2ydy=0$

Also from $f=ax^2+2hxy+by^2-1$

$$df=2ax\,dx+2h(ydx+xdy)+2bydy=0$$

So we get two equations

$$xdx+ydy=0 \qquad \cdots \qquad \cdots \qquad (1)$$

and
$$(ax+hy)dx+(by+hx)dy=0. \qquad \cdots \qquad (2)$$

Multiplying (2) by λ and adding with (1) and then equating co-efficients of dx and dy we get

$$x+\lambda(ax+hy)=0 \qquad \cdots \qquad \cdots \qquad (3)$$

$$y+\lambda(by+hx)=0. \qquad \cdots \qquad \cdots \qquad (4)$$

Multiplying these two equations by x and y respectively and adding

$$x^2+y^2+\lambda(ax^2+2hxy+by^2)=0$$

or, $u+\lambda=0$

$\therefore \quad \lambda=-u.$

$\therefore$ from (3) and (4) we get for maximum or minimum

$$x-u(ax+hy)=0$$
$$y-u(by+hx)=0$$

or, $(1-au)x-huy=0$

and $-uhx+(1-bu)y=0.$

$\therefore \quad \dfrac{1-au}{uh}=\dfrac{hu}{1-bu}$

or, $(h^2-ab)u^2+(a+b)u-1=0$

$\therefore \quad u=\dfrac{-(a+b)\pm\sqrt{(a+b)^2+4(h^2-ab)}}{2(h^2-ab)}$ is the extreme value of u.

Ex. 3. If $u=a^3x^2+b^3y^2+c^3z^2$ where $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ find the maximum or minimum value of u. [*C. H. 1964*]

$\because \quad u=a^3x^2+b^3y^2+c^3z^2$

and $f=\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}-1$

we get, $du=2a^3x\,dx+2b^3y\,dy+2c^3z\,dz=0$

and $df=-\dfrac{1}{x^2}dx-\dfrac{1}{y^2}dy-\dfrac{1}{z^2}dz=0$

Thus we get two equations

$$a^3x\,dx+b^3y\,dy+c^3z\,dz=0 \quad \cdots \quad (1)$$

and $\dfrac{1}{x^2}dx+\dfrac{1}{y^2}dy+\dfrac{1}{z^2}dz=0 \quad \cdots \quad \cdots \quad (2)$

Multiplying (2) by λ and adding with (1) and then equating co-efficients of dx, dy and dz we get

$$a^3x+\frac{\lambda}{x^2}=0, \quad b^3y+\frac{\lambda}{y^2}=0, \quad c^3z+\frac{\lambda}{z^2}=0. \cdots (3)$$

Multiplying these equations by x, y, z respectively and adding

$$(a^3x^2+b^3y^2+c^3z^2)+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=0$$

or, $u+\lambda=0$

$\therefore \quad \lambda=-u.$

So from (3) $a^3x-\frac{u}{x^2}=0$

$\therefore \quad a^3x^3=u$

Similarly, $b^3y^3=u$ and $c^3z^3=u$

$\therefore \quad ax=by=cz=k$ (suppose)

$\therefore \quad x=\frac{k}{a}, \quad y=\frac{k}{b}, \quad z=\frac{k}{c}$

Putting in $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

$\frac{1}{k}(a+b+c)=1 \qquad \therefore \quad k=a+b+c$

$\therefore$ for maximum or minimum

$$x=\frac{1}{a}(a+b+c), \quad y=\frac{1}{b}(a+b+c), \quad z=\frac{1}{c}(a+b+c).$$

And the maximum or minimum value of u

$$=a^3\left(\frac{a+b+c}{a}\right)^2+b^3\left(\frac{a+b+c}{b}\right)^2+c^3\left(\frac{a+b+c}{c}\right)^2$$

$$=(a+b+c)^3. \qquad \cdots \qquad \cdots \qquad (4)$$

Discrimination :

Differentiating $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ with respect to x, taking z as a function of x and y

$$-\frac{1}{x^2}-\frac{1}{z^2}\frac{\partial z}{\partial x}=0 \qquad \therefore \quad \frac{\partial z}{\partial x}=-\frac{z^2}{x^2}.$$

From $u=a^3x^2+b^3y^2+c^3z^2$

$$\frac{\partial u}{\partial x}=2a^3x+2c^3z\frac{\partial z}{\partial x}$$

$$=2a^3x+2c^3z\left(-\frac{z^2}{x^2}\right)$$

$$=2a^3x-\frac{2c^3z^3}{x^2}$$

$$\therefore \quad A=\frac{\partial^2 u}{\partial x^2}=2a^3-2c^3\left(-\frac{2z^3}{x^3}+\frac{3z^2}{x^2}\frac{\partial z}{\partial x}\right)$$

$$=2a^3+\frac{4c^3z^3}{x^3}+\frac{6c^3z^4}{x^4}$$

$$=2a^3+4c^3\cdot\frac{a^3}{c^3}+6c^3\left(\frac{a}{c}\right)^4 \qquad \left[\because \frac{z}{x}=\frac{a}{c}\right]$$

$$=6\left(a^3+\frac{a^4}{c}\right)$$

Similarly, $C=\dfrac{\partial^2 u}{\partial y^2}=6\left(b^3+\dfrac{b^4}{c}\right)$

Also, $B=\dfrac{\partial^2 u}{\partial x\partial y}=-\dfrac{2c^3}{x^2}\cdot 3z^2\dfrac{\partial z}{\partial y}$

$$=-\frac{2c^3}{x^2}\cdot 3z^2\left(-\frac{z^2}{y^2}\right)$$

$$=6c^3\left(\frac{z}{x}\right)^2\left(\frac{z}{y}\right)^2$$

$$=6c^3\left(\frac{a}{c}\right)^2\left(\frac{b}{c}\right)^2=\frac{6a^2b^2}{c}$$

Now $AC-B^2=36\left(a^3+\dfrac{a^4}{c}\right)\left(b^3+\dfrac{b^4}{c}\right)-\left(\dfrac{6a^2b^2}{c}\right)^2$

$$=\frac{36a^2b^2}{c^2}\left\{abc(a+b+c)\right\}.$$

>0 if $abc(a+b+c)>0$.

Thus if a, b, c are all positive

$AC-B^2>0$ and $A>0$ and hence u is minimum. The minimum value is given by (4).

Ex. 4. Find the maximum value of $f(x, y, z)=x^2y^2z^2$ subject to the condition $x^2+y^2+z^2=c^2$. *(C. H. 1960)*

Let $u=f(x, y, z)=x^2y^2z^2$

and $f_1=x^2+y^2+z^2-c^2$

then $\log u=2 \log x+2 \log y+2 \log z$

$$\therefore \quad \frac{1}{u}du=\frac{2}{x}dx+\frac{2}{y}dy+\frac{2}{z}dz=0$$

and $df_1=2xdx+2ydy+2zdz=0$

So we get two equations

$$\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0 \quad \cdots \quad \cdots \quad (1)$$

and $\quad xdx+ydy+zdz=0 \quad \cdots \quad \cdots \quad (2)$

Multiplying (2) by λ and adding with (1) and then equating coefficient of dx, dy and dz, we have

$$\frac{1}{x}+\lambda x=0, \quad \frac{1}{y}+\lambda y=0, \quad \frac{1}{z}+\lambda z=0.$$

Multiplying these equations by x, y, z respectively and adding

$$1+1+1+\lambda(x^2+y^2+z^2)=0$$

or, $\quad 3+\lambda c^2=0$

$$\therefore \quad \lambda=-\frac{3}{c^2}.$$

So from $\frac{1}{x}+\lambda x$ we get $\frac{1}{x}-\frac{3x}{c^2}=0$

or, $\quad 3x^2=c^2 \quad$ or, $\quad x^2=\frac{1}{3}c^2.$

Similarly, $\quad y^2=z^2=\frac{1}{3}c^2.$

$\therefore$ The maximum or minimum value of u

$$=\frac{1}{3}c^2 \cdot \frac{1}{3}c^2 \cdot \frac{1}{3}c^2=\frac{1}{27}c^6.$$

Discrimination :

$$u=x^2y^2(c^2-x^2-y^2)=x^2y^2c^2-x^4y^2-x^2y^4$$

$$\therefore \quad \frac{\partial u}{\partial x}=2xy^2c^2-4x^3y^2-2xy^4$$

$$\therefore \quad A=\frac{\partial^2 u}{\partial x^2}=2y^2c^2-12x^2y^2-2y^4$$

$$=2\,.\,\frac{1}{3}c^2\,.\,c^2-12\,.\,\frac{1}{3}c^2\,.\,\frac{1}{3}c^2-2\,.\,\frac{1}{9}c^4$$

$$=-\frac{8c^4}{9}\,.$$

$$B=\frac{\partial^2 y}{\partial x\partial y}=4xyc^2-8x^3y-8xy^3$$

$$=4.\frac{1}{3}c^4-8xy(x^2+y^2)$$

$$=\frac{4}{3}c^4-8\,\frac{1}{3}c^2.\frac{2}{3}c^2$$

$$=-\frac{4}{9}c^4.$$

Also, $$\frac{\partial u}{\partial y}=2x^2yc^2-2x^4y-4x^2y^3$$

$$\therefore \quad C=\frac{\partial^2 u}{\partial y^2}=2x^2c^2-2x^4-12x^2y^2$$

$$=2.\frac{1}{3}c^4-2.\frac{1}{9}c^4-12.\frac{1}{9}c^4$$

$$=-\frac{8}{9}c^4$$

Now, $\because$ $$AC-B^2=\left(-\frac{8}{9}c^4\right)\left(-\frac{8}{9}c^4\right)-\left(-\frac{4}{9}c^4\right)^2$$

$$=\frac{64}{81}c^8-\frac{16}{81}c^8$$

$$=\frac{48}{81}c^8>0$$

And $A<0$, $f(x, y, z)$ is maximum and its maximum value $=\frac{1}{27}c^6$.

Ex. 5. Find the volume of the greatest rectangular parallelopiped inscribed in the ellipsoid whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Let Volume $V=(2x)(2y)(2z)=8xyz$

And $f=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$

$\therefore \quad dV=8(yzdx+xzdy+xydz)=0$

$$df=\frac{2x}{a^2}dx+\frac{2y}{b^2}dy+\frac{2z}{c^2}dz=0$$

$\therefore$ For stationary values, we have

$$yz+\lambda\frac{x}{a^2}=0$$

$$xz+\lambda\frac{y}{b^2}=0$$

$$xy+\lambda\frac{z}{c^2}=0$$

$$\therefore \quad 3xyz+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=0$$

or, $\quad 3xyz+\lambda=0$

$$\therefore \quad \lambda=-3xyz=-\frac{3V}{8}.$$

Now $\quad yz=-\frac{\lambda x}{a^2}$

or, $\quad xyz=-\frac{\lambda x^2}{a^2}$

or, $\quad \frac{V}{8}=\frac{3V}{8}\cdot\frac{x^2}{a^2}$

or, $\quad \frac{x^2}{a^2}=\frac{1}{3}.$

Similarly, $\frac{y^2}{b^2}=\frac{z^2}{c^2}=\frac{1}{3}$.

$\therefore$ maximum or minimum value of

$$V=8.\frac{a}{\sqrt{3}}.\frac{b}{\sqrt{3}}.\frac{c}{\sqrt{3}}=\frac{8abc}{3\sqrt{3}}.$$

Discrimination :

$\because$ $f=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$

$\therefore$ $\frac{2x}{a^2}+\frac{2z}{c^2}.\frac{\partial z}{\partial x}=0$

or, $\frac{\partial z}{\partial x}=-\frac{xc^2}{za^2}$.

Now from $V=8xyz$

$$\frac{\partial V}{\partial x}=8y\left(z+x\frac{\partial z}{\partial x}\right)=8y\left(z-\frac{x^2c^2}{za^2}\right)$$

$$\frac{\partial^2 V}{\partial x^2}=8y\left\{\frac{\partial z}{\partial x}-\frac{c^2}{a^2}\left(\frac{2xz-\frac{\partial z}{\partial x}.x^2}{z^2}\right)\right\}$$

$$=8y\left\{-\frac{xc^2}{za^2}-\frac{c^2}{a^2z^2}\left(2xz+\frac{xc^2.x^2}{za^2}\right)\right\}$$

$$=-ve \text{ when } \frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{1}{\sqrt{3}}.$$

Hence, volume is maximum and maximum volume$=\frac{8abc}{3\sqrt{3}}$

Ex. 6. Prove that of all rectangular parallelopiped of same volume, the cube has the least surface.

If V be the volume and S be the surface of a parallelopiped of dimensions x, y, z, then

$$V=xyz \text{ and } S=2(xy+yz+zx).$$

$\therefore$ $dS=2(y+z)dx+2(z+x)dy+2(x+y)dz=0$

$dV=yzdx+zxdy+xydz=0.$

$\therefore$ For stationary points

$(y+z)+\lambda yz=0$

$$(z+x)+\lambda zx=0$$
$$(x+y)+\lambda xy=0.$$

or, $\dfrac{y+z}{yz}=\dfrac{z+x}{zx}=\dfrac{x+y}{xy}=-\lambda$

or, $\dfrac{1}{z}+\dfrac{1}{y}=\dfrac{1}{x}+\dfrac{1}{z}=\dfrac{1}{y}+\dfrac{1}{x}$

$\therefore\quad x=y=z.$

Now $\because\quad xyz=V$

$\therefore\quad x^3=V \quad \therefore\quad x=V^{\frac{1}{3}}$

$\therefore\quad x=y=z=V^{\frac{1}{3}}$

Again from $xyz=V$, differentiating *w. r.* to x and y regarding z as a function of x and y

$$yz+xy\frac{\partial z}{\partial x}=0 \qquad \therefore\quad \frac{\partial z}{\partial x}=-\frac{z}{x}$$

and $$xz+xy\frac{\partial z}{\partial y}=0 \qquad \therefore\quad \frac{\partial z}{\partial y}=-\frac{z}{y}.$$

Now from $S=2(xy+yz+zx)$

$$\frac{\partial S}{\partial x}=2\left(y+y\frac{\partial z}{\partial x}+z+x\frac{\partial z}{\partial x}\right)$$

$$=2y-2y\frac{z}{x}+2z-2x\frac{z}{x}$$

$$=2y-2\frac{yz}{x}.$$

$$\therefore\quad \frac{\partial^2 S}{\partial x^2}=-2y\left\{\frac{x\dfrac{\partial z}{\partial x}-z}{x^2}\right\}=-\frac{2y}{x^2}\left\{-\frac{z}{x}\cdot x-z\right\}$$

$$=\frac{4yz}{x^2}=4 \quad \text{at } x=y=z.$$

Similarly, $\dfrac{\partial^2 S}{\partial y^2}=4$ and $\dfrac{\partial^2 S}{\partial x\,\partial y}=2$

Since, $AC-B^2=4\times4-(2)^2=12>0$ and $A>0$

$\therefore\quad S$ is least when $x=y=z$.

Ex. 7. Find the maximum value of $x^m y^n$ if $x+y=k$, a constant, the quantities being all positive.

Hence, show that

$$a^m b^n < \left(\frac{ma+nb}{m+n}\right)^{m+n},$$ **except when $a=b$** (*C. H.* 1965)

Let $u=x^m y^n$ and $f=x+y-k=0$

Then $\log u = m\log x + n\log y$

or, $\frac{1}{u}du=\frac{m}{x}dx+\frac{n}{y}dy=0$

and $df=dx+dy=0$

So we get two equations

$$\frac{m}{x}dx+\frac{n}{y}dy=0$$

and $dx+dy=0$

$\therefore$ For stationary points

$$\frac{m}{x}+\lambda=0 \quad \text{and} \quad \frac{n}{y}+\lambda=0 \qquad \cdots \quad (1)$$

Multiplying (1) by x and y respectively and adding

$$m+n+\lambda(x+y)=0$$

or, $m+n+\lambda k=0$

$$\therefore \quad \lambda=-\frac{m+n}{\;}$$

$\therefore$ From (1) $x=-\frac{m}{\lambda}=\frac{mk}{m+n}$

$$y=-\frac{n}{\lambda}=\frac{nk}{m+n}.$$

$\therefore$ Maximum or minimum value of u

$$=x^m \cdot y^n$$

$$=\left(\frac{mk}{m+n}\right)^m \cdot \left(\frac{nk}{m+n}\right)^n$$

$$=m^m \cdot n^n\left(\frac{k}{m+n}\right)^{m+n}$$

Discrimination.

Differentiating $x+y=k$, treating y as a function of x

$$1+\frac{\partial y}{\partial x}=0 \qquad \therefore \quad \frac{\partial y}{\partial x}=-1$$

Again from $u=x^m y^n$,

$$\log u=m \log x+n \log y$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x}=\frac{m}{x}+\frac{n}{y}\frac{\partial y}{\partial x}=\frac{m}{x}-\frac{n}{y}$$

Differentiating again w. r. to x

$$-\frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{u}\frac{\partial^2 u}{\partial x^2}=-\frac{m}{x^2}+\frac{n}{y^2}\frac{\partial y}{\partial x}$$

$$=-\frac{m}{x^2}-\frac{n}{y^2}$$

or, $$\frac{1}{u}\frac{\partial^2 u}{\partial x^2}=-\left(\frac{m}{x^2}+\frac{n}{y^2}\right)<0$$

$\left[\because \text{ for extreme points } \frac{\partial u}{\partial x}=0.\right]$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}<0$$

Hence u is maximum

Deduction.

Maximum value of $x^m y^n=m^m n^n\left(\frac{k}{m+n}\right)^{m+n}$

or, Maximum value of $\left(\frac{x}{m}\right)^m\left(\frac{y}{n}\right)^n=\left(\frac{k}{m+n}\right)^{m+n}$ $\quad \cdots \quad$ (1)

If we put $\frac{x}{m}=a$ and $\frac{y}{n}=b$

Then $x=am$ and $y=bn$. and so $am+bn=x+y=k$

$\therefore$ from (1) we get

maximum value of $a^m b^n=\left(\frac{am+bn}{m+n}\right)^{m+n}$ except when $a=b$

Exercise 6

1. Show that the maximum value of $u=x^3y^2(1-x-y)$ is at $x=\frac{1}{2}$, $y=\frac{1}{3}$.

2. Prove that the minimum value of $u=xy+c^3\left(\frac{1}{x}+\frac{1}{y}\right)$ is $3c^2$.

3. Show that $x^2y^2-5x^2-8xy-5y^2$ is maximum at the origin.

4. Find the stationary points of the function

$$x^3+3xy^2-15x^2-15y^2+72x$$

examining whether they are maximum or minimum. (D. H. 1952)

5. Find the maximum value of $\cos x . \cos y . \cos z$, where x, y, z are the angles of a plane triangle. [*Ans.* 1/8] (C. H. 1959)

6. Show that $u=\sin x \sin y \sin (x+y)$ is maximum at $x=y=\pi/3$ and the maximum value of u is $3\sqrt{3}/8$.

7. Show that the minimum value of $x+y+z$ subject to the condition $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$ is $(\sqrt{a}+\sqrt{b}+\sqrt{c})^2$.

8. Let $u=x^2+y^2+z^2$ subject to the condition $ax+by+cz=p$. Show that the minimum value of u is $p^2/(a^2+b^2+c^2)$.

9. If $u=x^2+y^2+z^2$ where $ax^2+by^2+cz^2=1$ show that the extreme values of u are the roots of the equation $(1-au)(1-bu)(1-cu)=0$.

CHAPTER VII

MULTIPLE INTEGRALS

7·1. Double Integral.

Let us consider a function $f(x, y)$ which is bounded and defined in a region A on a plane. Let the region A be subdivided into sub-regions $A_1, A_2, \cdots A_n$ in any manner, and let w_r be the area of the sub-region A_r. Let us now construct the two sums S and s as follows

$$S=\sum_{r=1}^{n} w_r M_r, \quad s=\sum_{r=1}^{n} w_r m_r$$

where M_r and m_r are the bounds of $f(x, y)$ in A_r.

If $f(x, y)$ is continuous in the region A, then S and s tend to the limit I as the area of the sub-regions w_r tends to zero, and in that case $f(x, y)$ is said to possess a double integral over the field A and is denoted by

$$\iint f(x, y) dA \quad or, \quad \iint f(x, y)\, dx.dy.$$

$$\text{Hence } I = Lt \sum w_r M_r = Lt \sum w_r m_r$$
$$= \iint_A f(x, y) dx\, dy.$$

7·2. Geometrical meaning of the double Integral.

Suppose the double integral $\iint_A f(x, y) dx\, dy$ exists over the field A. Let the region A be subdivided into sub-regions $A_1, A_2, \cdots A_n$ in any manner and let w_r be the area of the sub-region A_r. Let us now construct a prism on w_r as base and with sides parallel to z-axis cut off an area A_r'

from the surface $z=f(x, y)$. If V_r be the volume of this prism, we can write

$$\sum_1^n w_r m_r \leqslant \sum_1^n V_r \leqslant \sum_1^n w_r M_r.$$

or, $$Lt \sum w_r m_r \leqslant Lt\, \Sigma V_r \leqslant Lt\, \Sigma w_r M_r$$

Since $\iint_A f(x, y)dxdy$ exists

$$\iint_A f(x, y)dxdy = Lt \Sigma w_r m_r = Lt \Sigma w_r M_r$$

Hence $\iint f(x, y)dxdy = \Sigma V_r = V$

where V is the volume of the space generated by a line parallel to the z axis and moving along the contour of the region A and bounded by the region A below and the surface $z=f(x, y)$ above.

7·3. Theorems on the Evolution of Double Integrals.

I. If f(x, y) be a continuous function defined in the region A and having a double integral over the field A, then

$$\iint_R \mathbf{f(x, y)dxdy} = \int_{x_0}^{X} \mathbf{dx} \int_{y_0}^{Y} \mathbf{f(x, y)dy}$$

when A is the rectangular region R.

Proof. Let the rectangular region R where $f(x, y)$ is continuous be bounded by the lines $x=x_0$, $x=X$ and $y=y_0$, $y=Y$ where x_0, y_0, X, Y are all constants. If the rectangle R be now divided into sub-rectangles $R_1, R_2, \cdots R_n$ by drawing lines parallel to x and y axes, then *a* small rectangle R_{ik} is formed bounded by the lines $x=x_{i-1}$, $x=x_i$; $y=y_{k-1}$, $y=y_k$.

Let w_{ik} be the area of the rectangle R_{ik},

then $w_{ik}=(x_i-x_{i-1})(y_k-y_{k-1})$ ... ... (1)

Let (ξ_{ik}, η_{ik}) be a point in R_{ik} and M_{ik}, m_{ik} are the upper and lower bounds of $f(x, y)$ in it. Then clearly

$$m_{ik}w_{ik} \leqslant f(\xi_{ik}, \eta_{ik})w_{ik} \leqslant M_{ik}w_{ik}$$

or, $$\sum_{i=1}^{n}\sum_{k=1}^{n} m_{ik}w_{ik} \leqslant \sum_{i=1}^{n}\sum_{k=1}^{m} f(\xi_{ik}, \eta_{ik})w_{ik} \leqslant \sum_{i=1}^{n}\sum_{k=1}^{m} M_{ik}w_{ik}$$

or, $$Lt \sum_{i=1}^{n}\sum_{k=1}^{m} m_{ik}w_{ik} \leqslant Lt \sum_{i=1}^{n}\sum_{k=1}^{m} f(\xi_{ik}, \eta_{ik})w_{ik}$$

$$\leqslant Lt \sum_{i=1}^{n}\sum_{k=1}^{m} M_{ik}w_{ik} \quad \cdots \quad (2)$$

But, since, the double integral exists

$$\iint_R f(x, y)dxdy = Lt \sum_{i=1}^{n}\sum_{k=1}^{m} m_{ik}w_{ik} = Lt \sum_{i=1}^{n}\sum_{k=1}^{m} M_{ik}w_{ik}$$

Hence from (2)

$$\iint_R f(x, y)dxdy = Lt \sum_{i=1}^{n}\sum_{k=1}^{m} f(\xi_{ik}, \eta_{ik})w_{ik}$$

$$= Lt \sum_{i=1}^{n}\sum_{k=1}^{m} f(\xi_{ik}, \eta_{ik})(x_i - x_{i-1})(y_k - y_{k-1}) \text{ from (1)} \cdots (3)$$

Let S_r denotes the sum of all the rectangles between the lines $x=x_{i-1}$ and $x=x_i$. Then

$$S_r=(x_i-x_{i-1})[f(\xi_{i1}, \eta_{i1})(y_1-y_0)+f(\xi_{i2}, \eta_{i2})(y_2-y_1)+\cdots$$
$$\cdots\cdots+f(\xi_{ik}, \eta_{ik})(y_k-y_{k-1})+\cdots\cdots]$$

Let $\xi_{i1}=\xi_{i2}=\cdots=\xi_{ik}=x_{i-1}$(*a* constant quantity) then

$$S_r=(x_i-x_{i-1})[f(x_{i-1}, \eta_{i1})(y_1-y_0)+f(x_{i-1}, \eta_{i2})(y_2-y_1)+$$
$$\cdots\cdots+f(x_{i-1}, \eta_{ik})(y_k-y_{k-1})+\cdots \quad \cdots \quad (4)$$

Let us now choose $\eta_{i1}, \eta_{i2} \cdots, \eta_{ik} \cdots$ in such a way that

$$f(x_{i-1}, \eta_{i1})(y_1 - y_0) + f(x_{i-1}, \eta_{i2})(y_2 - y_1) = \cdots\cdots$$

$$= \int_{y_0}^{Y} f(x_{i-1}, y)dy \text{ where } x_{i-1} \text{ is a constant.}$$

Then from (4)

$$S_r = (x_i - x_{i-1}) \int_{y_0}^{Y} f(x_{i-1}, y)dy.$$

Now considering all such sums of rectangles between the parallel lines $(x = x_0, x = x_1)$, $(x = x_1, x = x_2)$, $(x = x_2, x = x_3) \cdots$ and adding we get the total area of the rectangles.

Hence from (3) we get

$$\iint f(x, y)dxdy = Lt\Big\{(x_1 - x_0) \int_{y_0}^{Y} f(x_0, y)dy +$$

$$(x_2 - x_1) \int_{y_0}^{Y} f(x_1, y)dy + \cdots\Big\}$$

$$= Lt\{(x_1 - x_0)\phi(x_0) + (x_2 - x_1)\phi(x_1) + \cdots$$

$$\cdots + (x_i - x_{i-1})\phi(x_{i-1}) + \cdots\}.$$

$$\left[\text{putting } \phi(x) = \int_{y_0}^{Y} f(x, y)dy.\right]$$

$$= Lt\{\delta_0\phi(x_0) + \delta_1\phi(x_1) + \cdots + \delta_r\phi(x_r) + \cdots\}$$

$$= Lt \sum_{r=0}^{n} \delta_r\phi(x_r)$$

$$= \int_{x_0}^{X} \phi(x)dx \text{ in the limit as the greatest } \delta r \to 0 \text{ and}$$

assuming $\phi(x)$ is continuous in the interval.

$$= \int_{x_0}^{X} dx \int_{y_0}^{Y} f(x, y)\, dy$$

Note. The above result gives us a method of evoluting a double integral. Thus to evolute the double integral $f(x, y)$, $f(x, y)$ may be integrated first with respect to y between the limits y_0 and Y keeping x constant, then the resulting function which is a function of x alone may be integrated between the limits x_0 and X.

II. If f(x, y) be a continuous function defined in a region A and having a double integral over the field A, then

$$\iint_R \mathbf{f(x, y)\, dxdy} = \int_{y_0}^{Y} \mathbf{dy} \int_{x_0}^{X} \mathbf{f(x, y)\, dx}$$

where A is the rectangular region R.

Proof. As in the last theorem, from (3)

$$\iint_R f(x, y)\, dx\, dy = Lt \sum_{i=1}^{n} \sum_{k=1}^{m} f(\xi_{ik}, \eta_{ik})(x_i - x_{i-1})(y_k - y_{k-1}) \quad \cdots \quad (3)$$

Let S_r denote the sum of all the rectangles between the lines $y = y_{k-1}$ and $y = y_k$, then

$$S_r = (y_k - y_{k-1})[f(\xi_{i1}, \eta_{i1})(x_1 - x_0) + f(\xi_{i2}, \eta_{i2})(x_2 - x_1) + \cdots \\ \cdots + f(\xi_{ik}, \eta_{ik})(x_i - x_{i-1}) + \cdots]$$

Let $\eta_{i1} = \eta_{i2} = \cdots\cdots \eta_{ik} = y_{k-1}$ (a constant quantity) then

$$S_r = (y_k - y_{k-1})[f(\xi_{i1}, y_{k-1})(x_1 - x_0) + f(\xi_{i2}, y_{k-1})(x_2 - x_1) + \\ \cdots + f(\xi_{ik}, y_{k-1})(x_i - x_{i-1}) + \cdots] \quad \cdots \quad (4)$$

Now choose $\xi_{i1}, \xi_{i2}, \cdots \xi_{ik} \cdots$ in such a way that

$$f(\xi_{i1}, y_{k-1})(x_1 - x_0) + f(\xi_{i2}, y_{k-1})(x_2 - x_1) + \cdots$$

$$= \int_{x_0}^{X} f(x, y_{k-1})\, dx, \text{ where } y_{k-1} \text{ is a constant.}$$

Then from (4)

$$S_r=(y_k-y_{k-1})\int_{x_0}^{X} f(x,\ y_{k-1})dx$$

Now considering all such sums of rectangles between the parallel lines $(y=y_0,\ y=y_1)$, $(y=y_1,\ y=y_2)$, $(y=y_2,\ y=y_3)\cdots$ and adding we get the total sum of rectangles *i.e.*,

$$\sum_{i=1}^{n}\sum_{k=1}^{m} f(\xi_{ik},\ \eta_{ik})(x_i-x_{i-1})(y_k-y_{k-1})$$

Hence from (3) we get

$$\iint_R f(x,\ y)\ dx\ dy$$

$$=Lt\Big[(y_1-y_0)\int_{x_0}^{X} f(x,y_0)dx+(y_2-y_1)\int_{x_0}^{X} f(x,\ y_1)dx + \cdots\Big]$$

$$=Lt\{(y_1-y_0)\phi(y_0)+(y_2-y_1)\phi(y_1)+\cdots$$
$$\cdots+(y_k-y_{k-1})\phi(y_{k-1})+\cdots\}$$

putting $\phi(y)=\int_{x_0}^{X} f(x,\ y)dx$

$$=Lt\ \{\delta_0\phi(y_0)+\delta_1\phi(y_1)+\cdots+\delta_r\phi(y_r)+\cdots$$

$$=Lt\sum_{r=0}^{n}\delta_r\phi(y_r)$$

$=\int_{y_0}^{Y}\phi(y)dy$. assuming $\phi(y)$ is continuous in the range considered and greatest of δ_r tending to zero.

$$=\int_{y_0}^{Y} dy\int_{x_0}^{X} f(x,\ y)\ dx$$

Note. The second method of evolution of a double integral is that we may first integrate $f(x, y)$ with respect to x between the limits x_0 and X and the resulting function which is a function of y alone may then be integrated with respect to y between the limits y_0 and Y.

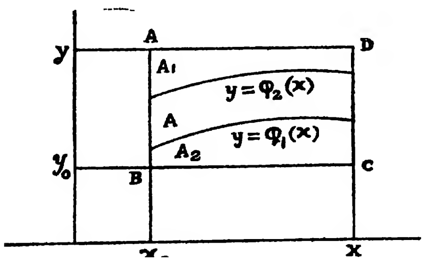
Cor. From the above two theorems, it follows that

$$\int_{x_0}^{X} dx \int_{y_0}^{Y} f(x, y)\, dy = \int_{y_0}^{Y} dy \int_{x_0}^{X} f(x, y) dx \quad \text{over the field } A,$$

where $f(x, y)$ is continuous in A and x_0, y_0, X, Y are all constants.

III. Let a function $f(x, y)$ be continuous in a region A which is a plane surface bounded by $x=x_0$, $x=X$, $y=\phi_1(x)$, $y=\phi_2(x)$ and that any line parallel to y-axis meets the bounding curves in two points only. If then $\phi_1(x)$, $\phi_2(x)$ are continuous functions of x in (x_0, X) and $\phi_1(x) \leqslant \phi_2(x)$, then

$$\iint_A f(x, y)\, dx dy = \int_{x_0}^{X} dx \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy.$$



Let R be the rectangular region $ABCD$ bounded by $x=x_0$, $x=X$, $y=y_0$, $y=Y$. Also, let A be the region entirely lying in R bounded by $x=x_0$, $x=X$, $y=\phi_1(x)$ and $y=\phi_2(x)$.

Then clearly $R = A + A_1 + A_2$.

Now consider a new function $F(x, y)$ defined as

(1) $F(x, y)=0$ for any point in A_1 and A_2

(2) $F(x, y)=f(x, y)$ for any point in A or on its contour.

Then

$$\iint_R F(x, y)dxdy=\iint_A F(x, y)dxdy+\iint_{A_1} F(x, y)dxdy$$
$$+\iint_{A_2} F(x, y)dxdy.$$
$$=\iint_A f(x, y)\,dxdy. \quad \cdots \quad \cdots \quad (1)$$

Now the field of the L.H.S. being rectangular

$$\iint_R F(x, y)dxdy=\int_{x_0}^{X} dx\int_{y_0}^{Y} F(x, y)dy$$

$\therefore$ From (1)

$$\int_{x_0}^{X} dx\int_{y_0}^{Y} F(x, y)dy=\iint_A f(x, y)dx.dy \quad \cdots \quad (2)$$

Now
$$\int_{y_0}^{Y} F(x, y)dy=\int_{y_0}^{y=\phi_1(x)} F(x, y)dy+\int_{\phi_1(x)}^{\phi_2(x)} F(x, y)dy$$
$$+\int_{\phi_2(x)}^{Y} F(x, y)dy.$$
$$=0+\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy+0$$
$$=\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy$$

Hence putting in (2) we get

$$\iint_A f(x, y)dxdy = \int_{x_0}^{X} dx \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy.$$

Cor. Similarly, as in III.

$$\iint_A f(x, y)dxdy = \int_{y_0}^{Y} dy \int_{\phi_1(y)}^{\phi_2(y)} f(x, y)dx.$$

where A is region bounded by $y=y_0$, $y=Y$, $x=\phi_1(y)$ and $x=\phi_2(y)$ and $f(x, y)$ being continuous in the region considered.

IV. If a continuous function f(x, y), $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ be of the form $f(x, y)=\phi(x)\psi(y)$, then

$$\iint_R f(x, y)dxdy = \int_a^b \phi(x)dx.\int_c^d \psi(y)dy.$$

(C. H. 1960, 63)

As in theorem (I), from (3)

$$\iint_R f(x, y)dxdy = Lt\sum_{i=1}^{n}\sum_{k=1}^{m} f(\xi_{ik}, \eta_{ik})(x_i - x_{i-1})(y_k - y_{k-1})$$

$$= Lt\sum_{i=1}^{n}\sum_{k=1}^{m} \phi(\xi_{ik})\psi(\eta_{ik})(x_i - x_{i-1})(y_k - y_{k-1}) \quad \cdots \quad (1)$$

Let S_r denotes the sum of all the rectangles between the lines $x=x_{i-1}$ and $x=x_i$, then

$$S_r = (x_i - x_{i-1})[\phi(\xi_{i1})\psi(\eta_{i1})(y_1 - y_0) + \phi(\xi_{i2})\psi(\eta_{i2})(y_2 - y_1) \\ \cdots + \phi(\xi_{ik})\psi(\eta_{ik})(y_k - y_{k-1}) + \cdots]$$

Let $\xi_{i1} = \xi_{i2} = \cdots = \xi_{ik} = x_{i-1}$ (constant),

Then $S_r = (x_i - x_{i-1})\phi(x_{i-1})[\psi(\eta_{i1})(y_1 - y_0)$
$+ \psi(\eta_{i2})(y_2 - y_1) + \cdots \quad \cdots(1)$

We can choose η_{i1}, η_{i2}, $\eta_{i3}\cdots$in such away that in the limit

$$\psi(\eta_{i1})(y_1-y_0)+\psi(\eta_{i2})(y_2-y_1)+\cdots$$
$$=\int_{y_0}^{Y}\psi(y)dy.$$

So from (1)

$$S_i=(x_i-x_{i-1})\phi(x_{i-1})\int_{y_0}^{Y}\psi(y)dy.$$

Now considering all such sums of rectangles between the parallel lines $(x=x_0,\ x=x_1)$, $(x=x_1,\ x=x_2)$ etc. and adding we get the total area of the rectangles.

So from (1) we get

$$\iint_R f(x,y)dxdy=\int_{y_0}^{Y}\psi(y)dy\ .\ \Big[Lt\Big\{(x_1-x_0)\phi(x_0)$$
$$+(x_2-x_1)\phi(x_1)+\cdots+(x_i-x_{i-1})\phi(x_{i-1})+\cdots\Big\}\Big]$$
$$=\int_{y_0}^{Y}\psi(y)dy.\ \int_{x_0}^{X}\phi(x)dx$$

Hence, replacing x_0, X, y_0 and Y by a, b, c and d respectively we get

$$\iint_R f(x,y)dx=\int_a^b\phi(x)dx\int_c^d\psi(y)dy.$$

7.4. Triple Integrals or Volume Integrals.

Let us consider a function $f(x, y, z)$ which is bounded and defined in a three dimensional region $V[a\leqslant x\leqslant b,$ $c\leqslant y\leqslant d,\ e\leqslant z\leqslant f]$

Let the intervals (a, b), (c, d), (e, f) be divided into p, q, r parts respectively by the points

$$a=x_0, x_1, x_2, \cdots, x_{i-1}, x_i, \cdots x_p=b$$
$$c=y_0, y_1, y_2, \cdots, y_{i-1}, y_i, \cdots y_q=d$$
$$e=z_0, z_1, z_2, \cdots, z_{i-1}, z_i, \cdots z_r=f$$

and consider the ijkth volume denoted by V_{ijk} where i varies from 1 to p, j varies from 1 to q and k varies from 1 to r respectively.

Then $V_{ijk}=(x_i-x_{i-1})(y_j-y_{j-1})(z_k-z_{k-1})$

Let m_{ijk} and M_{ijk} be the lower and upper bound of $f(x, y, z)$ in V_{ijk} and construct the upper and lower sums S and s as,

$$S=\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{r}(M_{ijk}V_{ijk})$$

$$s=\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{r}(m_{ijk}V_{ijk})$$

If (x, y, z) be any point in V_{ijk}, then evidently

$$m_{ijk}\leqslant f(x, y, z)\leqslant M_{ijk}$$

or, $$m_{ijk}V_{ijk}\leqslant f(x, y, z)V_{ijk}\leqslant M_{ijk}V_{ijk}$$

or, $$\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{r}m_{ijk}V_{ijk}\leqslant \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{r}f(x, y, z)V_{ijk}$$

$$\leqslant \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{r}M_{ijk}V_{ijk}$$

or, $$s\leqslant \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{r=1}^{r}f(x, y, z)V_{ijk}\leqslant S$$

In the limit if S and s tend to the same limit, then $f(x, y, z)$ is said to be integrable over V and is usually written as

$$\iiint_V f(x, y, z)dxdydz.$$

So we write

$$\iiint_V f(x, y, z)dxdydz = Lt\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{r} f(x, y, z)V_{ijk}.$$

Also as in the case of double integral we can write

$$\iiint_V f(x, y, z)dxdydz = \int_e^f dz\int_c^d dy\int_a^b f(x, y, z)dx$$

$$= \int_c^d dy\int_e^f dz\int_a^b f(x, y, z)dx$$

$$= \int_c^d dy\int_a^b f(x, y, z)dx\int_e^f dz.$$

i e., the order of integration may be interchanged in any maner we like.

7.5. Change of Variables by Jacobians.

Let the variables $x_1, x_2, \cdots x_n$ be functions of $u_1, u_2, \cdots u_n$. Then the jacobians of these variables with respect to $u_1, u_2, \cdots u_n$ is the determinant

$$\begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} \cdots & \frac{\partial x_n}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} \cdots & \frac{\partial x_n}{\partial u_2} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial u_n} & \frac{\partial x_2}{\partial u_n} \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix}$$

This is shortly denoted by J or by $\dfrac{\partial(x_1, x_2, \cdots x_n)}{\partial(u_1, u_2, \cdots u_n)}$.

Thus if $x=f(u_1, u_2)$ $y=\phi(u_1, u_2)$

$$\text{Then } J=\frac{\partial(x, y)}{\partial(u_1, u_2)}=\begin{vmatrix} \dfrac{\partial x}{\partial u_1} & \dfrac{\partial y}{\partial u_1} \\ \dfrac{\partial x}{\partial u_2} & \dfrac{\partial y}{\partial u_2} \end{vmatrix}$$

Also if $x=f(u_1, u_2, u_3)$, $y=\phi(u_1, u_2, u_3)$, $z=\psi(u_1, u_2, u_3)$

$$\text{Then } J=\frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)}=\begin{vmatrix} \dfrac{\partial x}{\partial u_1} & \dfrac{\partial y}{\partial u_1} & \dfrac{\partial z}{\partial u_1} \\ \dfrac{\partial x}{\partial u_2} & \dfrac{\partial y}{\partial u_2} & \dfrac{\partial z}{\partial u_2} \\ \dfrac{\partial x}{\partial u_3} & \dfrac{\partial y}{\partial u_3} & \dfrac{\partial z_3}{\partial u_3} \end{vmatrix}$$

Suppose, we now require to transform the double integral $\iint_A f(x, y)dxdy$ to the variables u and v by the substitution $x=\phi(u, v)$, $y=\psi(u, v)$. Then it can be shown

$$\iint_A f(x, y)dxdy=\iint_{A'} f\{\phi(u, v), \psi(u, v)\} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv.$$

where A' is the region of the transformed integral in the (u, v) plane corresponding to the region A in the (x, y) plane.

Similarly, if $x=f(\xi, \eta, \zeta)$, $y=\phi(\xi, \eta, \zeta)$, $z=\psi(\xi, \eta, \zeta)$

$$\text{Then } \iiint_V F(x, y, z)dxdydz$$

$$=\iiint_{V'} F\{f(\xi, \eta, \zeta)\phi(\xi, \eta, \zeta)\psi(\xi, \eta, \zeta)\} \left| \frac{\partial(x, y, z)}{\partial(\xi, \eta, \zeta)} \right| d\xi d\eta d\zeta.$$

where V' is the region of the transformed integral corresponding to the original region V.

Illustrative Examples :

Ex. 1. Evaluate the integral

$$\iint \mathbf{x^2y^2dxdy} \text{ on } \mathbf{x^2+y^2 \leq 1.}$$

Here $f(x, y)=x^2y^2$.

Let $x=r\cos\theta$, $y=r\sin\theta$, so that $x^2+y^2=r^2$

Now $\iint x^2y^2dxdy$ on $x^2+y^2 \leq 1$

$$=\iint_{r\leq 1} f(r\cos\theta, r\sin\theta)\ \frac{\partial(x, y)}{\partial(r, \theta)}drd\theta$$

$$=\iint_{r\leq 1} r^2\cos^2\theta \,.\, r^2\sin^2\theta \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} drd\theta.$$

$$=\iint_{r\leq 1} r^4(\sin\theta\cos\theta)^2 \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} drd\theta$$

$$=\iint_{r\leq 1} r^4(\sin\theta\cos\theta)^2 .\, rdrd\theta.$$

$$=\int_0^1 r^5dr\int_0^{2\pi}(\sin\theta\cos\theta)^2\, d\theta.$$

$$=\frac{1}{6}\cdot\frac{1}{4}\int_0^{2\pi}\sin^2 2\theta\, d\theta$$

$$=\frac{1}{48}\int_0^{2\pi}2\sin^2 2\theta\, d\theta$$

$$=\frac{1}{48}\int_0^{2\pi}(1-\cos 4\theta)d\theta$$

$$=\frac{1}{48}\left[\theta-\frac{\sin 4\theta}{4}\right]_0^{2\pi}=\frac{\pi}{24}.$$

Ex. 2. Evaluate $\iint_R x^2y \, dxdy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (C. H. 1971)

Let $x=au$, $y=bv$, then R is transformed into R' where R' is the circle $u^2+v^2=1$.

$$\therefore \iint_R x^2ydxdy=\iint a^2u^2bv\frac{\partial(x, y)}{\partial(u, v)}dudv \text{ on } u^2+v^2=1$$

$$=a^2b\iint u^2v\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}dudv$$

$$=a^2b\iint u^2v\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}dudv$$

$$=a^3b^2\iint u^2vdudv \quad \text{on} \quad u^2+v^2=1$$

[Put $u=r\cos\theta$, $v=r\sin\theta$]

$$=a^3b^2\iint r^2\cos^2\theta.r\sin\theta\begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix}drd\theta \text{ on } r=1$$

$$=a^3b^2\iint r^3\sin\theta\cos^2\theta\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}drd\theta$$

$$=a^3b^2\iint r^4\sin\theta\cos^2\theta\, drd\theta$$

$$=a^3b^2\int_0^1 r^4\,dr\int_0^{\pi/2}\sin\theta(1-\sin^2\theta)d\theta$$

$$=\frac{1}{3}a^3b^2\int_0^1 r^4dr=\frac{1}{15}a^3b^2.$$

Ex. 3. **Evaluate** $\iint\limits_R \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dxdy$ **over the first quadrant of the ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. **(C. H. 1963, 72)**

Let $x=au$, $y=bv$, so that the region R is transformed into R' where R' is the circle $u^2+v^2=1$.

Then $\iint\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dxdy$ on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$=\iint(1-u^2-v^2)\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}dudv, \text{ on } u^2+v^2=1$$

$$=\iint(1-u^2-v^2)\begin{vmatrix}a & 0\\ 0 & b\end{vmatrix}dudv$$

$$=ab\iint(1-u^2-v^2)du\,dv$$

[Put $u=r\cos\theta$, $v=r\sin\theta$]

$$=ab\iint(1-r^2)\begin{vmatrix}\frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta}\\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta}\end{vmatrix}drd\theta, \quad \text{on } r=1$$

$$=ab\iint(1-r^2)\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}dr.d\theta$$

$$=ab\iint r(1-r^2)drd\theta$$

$$=ab\int_0^1 r(1-r^2)dr\int_0^{\pi/2}d\theta$$

$$=\frac{\pi ab}{2}\int_0^1(r-r^3)dr$$

$$=\frac{\pi ab}{8}.$$

Ex. 4. Evaluate $\iint\sqrt{\frac{a^2b^2-b^2x^2-a^2y^2}{a^4b^2+b^2x^2+a^2y^2}}dxdy$, the area of integration being the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (C. H. 1961, 68)

Let $x=au$, $y=bv$ so that the new region becomes the circle $u^2+v^2=1$.

$$\therefore \iint\sqrt{\frac{a^2b^2-b^2x^2-a^2y^2}{a^2b^2+b^2x^2+a^2y^2}}dxdy$$

$$=\iint\sqrt{\frac{1-u^2-v^2}{1+u^2+v^2}}\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} du\,dv \text{ on } u^2+v^2=1$$

$$=\iint\sqrt{\frac{1-u^2-v^2}{1+u^2+v^2}}\begin{vmatrix}a & 0\\ 0 & b\end{vmatrix} du\,dv$$

[Put $u=r\cos\theta$, $v=r\sin\theta$]

$$=ab\iint\sqrt{\frac{1-r^2}{1+r^2}}\begin{vmatrix}\frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta}\\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta}\end{vmatrix} du\,d\theta, \text{ on } r=1$$

$$=ab\iint\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr\,d\theta, \text{ on } r=1$$

$$=ab\int_0^{\pi/2} d\theta\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr \text{ [On the positive quadrant]}$$

$$=\frac{\pi ab}{2.2}\int_0^{\pi/2}\sqrt{\frac{1-\sin z}{1+\sin z}}\cos z\,dz \text{ putting } r^2=\sin z.$$

$$=\frac{\pi ab}{4}\int_0^{\pi/2}(1-\sin z)dz$$

$$=\frac{\pi ab}{8}(\pi-2).$$

Ex. 5. Evaluate the integral $\iint \sqrt{4a^2-x^2-y^2}\,dx\,dy$, taken over the upper half of the circle $x^2+y^2-2ax=0$.
(C. H. 1966)

Let $x=r\cos\theta$, $y=r\sin\theta$, so that the new region becomes the circle $r=2a\cos\theta$.

$$\therefore \quad \iint \sqrt{4a^2-x^2-y^2}\,dx\,dy$$

$$=\iint \sqrt{4a^2-r^2}\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix} dr\,d\theta$$

$$=\iint \sqrt{4a^2-r^2}\; r\,dr\,d\theta, \text{ on } r=2a\cos\theta$$

$$=\int_0^{\pi/2} d\theta \int_0^{2a\cos\theta} \sqrt{4a^2-r^2}\; r\,dr$$

$$=-\frac{1}{2}\int_0^{\pi/2} d\theta\int_{4a^2}^{4a^2\sin^2\theta} \sqrt{z}\,dz \qquad \left[\begin{array}{l}\text{Putting } 4a^2-r^2=z\\ \text{so that } -2r\,dr=dz.\end{array}\right]$$

$$=-\frac{1}{2}\int_0^{\pi/2} d\theta\;.\;\frac{2}{3}\Big[z^{\frac{3}{2}}\Big]_{4a^2}^{4a^2\sin^2\theta}$$

$$=-\frac{1.2}{2.3}\int_0^{\pi/2} d\theta.(8a^3\sin^3\theta-8a^3)$$

$$=\frac{8a^3}{3}\int_0^{\pi/2}(1-\sin^3\theta)d\theta=\frac{4a^3}{9}(3\pi-4).$$

Ex. 6. Evaluate the integral $\iint \sqrt{4x^2-y^2}\;dx\,dy$ extended over the triangle formed by the straight lines $y=0$ $x=1$, $y=x$.
(C. H. 1967, 70)

Draw the triangle OAB formed by $y=0$, $x=1$ and $y=x$, so that $OB=r$, and $\angle BOA=\theta$. Then $r\cos\theta=1$

$$\therefore \quad r=\frac{1}{\cos\theta}.$$

Now putting $x=r\cos\theta$, $y=r\sin\theta$

$$\iint \sqrt{4x^2-y^2}\,dx\,dy$$

$$=\iint \sqrt{4r^2\cos^2\theta-r^2\sin^2\theta}\;r\,dr\;d\theta \text{ on } r=\frac{1}{\cos\theta}.$$

$$=\int_0^{\pi/4}\int_0^{1/\cos\theta} \sqrt{4\cos^2\theta-\sin^2\theta}\,.\,r^2\,dr\,d\theta$$

$$=\int_0^{\pi/4} \sqrt{4\cos^2\theta-\sin^2\theta}\left[\frac{r^3}{3}\right]_0^{1/\cos\theta}d\theta.$$

$$=\frac{1}{3}\int_0^{\pi/4} \frac{\sqrt{4\cos^2\theta-\sin^2\theta}}{\cos^3\theta}\,d\theta.$$

$$=\frac{1}{3}\int_0^{\pi/4} \sqrt{4-\tan^2\theta}\,\sec^2\theta\,d\theta$$

$$=\frac{1}{3}\int_0^{1} \sqrt{2^2-z^2}\,dz \quad \text{putting } \tan\theta=z$$

$$=\frac{1}{3}\left[\frac{z\sqrt{2^2-z}}{2}+\frac{2^2}{2}\sin^{-1}\frac{z}{2}\right]_0^1=\frac{1}{18}(2\pi+3\sqrt{3}).$$

Ex. 7. Evaluate $\iiint xyz\,dx\,dy\,dz$, the volume of integration being the positive octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1$.

Let $x=au$, $y=bv$ and $z=cw$ so that the transferred field of integration becomes the sphere $u^2+v^2+w^2 \leqslant 1$.

$$\therefore \quad \iiint xyz\,dx\,dy\,dz=\iiint abc\;uvw\frac{\partial(x,y,z)}{\partial(u,v,w)}du\,dv\,dw.$$

$$=abc\iiint uvw \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} du.dv\,dw$$

$$=abc\iiint uvw \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} du\,dv\,dw$$

$$=a^2b^2c^2\iiint uvw\,du\,dv\,dw.$$

$$=a^2b^2c^2\iint uv\,du\,dv\int_0^{\sqrt{1-u^2-v^2}} w\,dw$$

$$=\frac{a^2b^2c^2}{2}\iint uv(1-u^2-v^2)du\,dv$$

$$=\frac{a^2b^2c^2}{2}\iint r^2\sin\theta\cos\theta(1-r^2)r\,dr\,d\theta$$

[transferring to polar]

$$=\frac{1}{2}a^2b^2c^2\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\int_0^1(r^3-r^5)\,dr$$

$$=\frac{1}{24}a^2b^2c^2\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta$$

$$=\frac{1}{48}a^2b^2c^2\int_0^{\pi/2}\sin\,2\theta\,d\theta=\frac{1}{48}a^2b^2c^2.$$

Ex. 8. Evaluate $\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$.

The field of integration being the positive octant of the sphere $x^2+y^2+z^2=1$.

Let $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$ and $z=r\cos\theta$. Then in the transformed field r varies from 0 to 1, θ and ϕ varies from 0 to $\pi/2$.

$$\therefore \iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$$

$$=\iiint \frac{1}{\sqrt{1-r^2}} \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}\, dr\, d\theta\, d\phi$$

$$=\iiint \frac{1}{\sqrt{1-r^2}} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix} dr\, d\theta\, d\phi$$

$$=\iiint \frac{1}{\sqrt{1-r^2}} \begin{vmatrix} \sin\theta\cos\phi, & \sin\theta\sin\phi, & \cos\theta \\ r\cos\theta\cos\phi, & r\cos\theta\sin\phi, & -r\sin\theta \\ -r\sin\theta\sin\phi, & r\sin\theta\cos\phi, & 0 \end{vmatrix} \begin{matrix} dr. \\ d\theta. \\ d\phi. \end{matrix}$$

$$=\iiint \frac{r^2\sin\theta}{\sqrt{1-r^2}}\, dr\, d\theta\, d\phi$$

$$=\int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\, d\theta \int_0^1 \frac{r^2\,dr}{\sqrt{1-r^2}} = \frac{\pi^2}{8}.$$

Ex. 9. Evaluate $\iiint (x^2+y^2+z^2)xyz\,dx\,dy\,dz$ **taken throughout the sphere $x^2+y^2+z^2 \leq 1$.** **(C. H. 1964)**

Let $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$ and $z=r\cos\theta$ so that in the transformed field r varies from 0 to 1, θ and ϕ varies from 0 to 2π.

$$\therefore \quad \iiint (x^2+y^2+z^2)xyz\,dx\,dy\,dz$$

$$=\iiint (r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi + r^2\cos^2\theta)$$

$$\times r^3 \sin^2\theta \cos\theta \sin\phi \cos\phi \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} dr.d\theta.d\phi$$

$$=\iiint r^2.r^3 \sin^2\theta \cos\theta \sin\phi \cos\phi.r^2 \sin\theta\,dr\,d\theta\,d\phi.$$

$$=\int_0^1 r^7\,dr \int_0^{2\pi} \sin^3\theta \cos\theta\,d\theta \int_0^{2\pi} \sin\phi \cos\phi\,d\phi.$$

$$=0 \quad \because \int^{2\pi} \sin^3\theta \cos\theta\,d\theta = 0.$$

Exercise 7

1. Evaluate $\iint x^2y^2\,dx\,dy$.

The region of integration being $x \geqslant 0$, $y \geqslant 0$ and $x^2+y^2 \leqslant 1$. Ans. $\pi/96$.

2. Evaluate $\iint \frac{dx\,dy}{(1+x^2+y^2)^2}$.

over a loop of the lemniscate $(x^2+y^2)^2-(x^2-y^2)=0$ (C. H. 1962)

3. Perfrom the integration

$$\iint \sqrt{2(ax+by)-(x^2+y^2)}\,dx\,dy.$$

taken over the circle

$$x^2+y^2-2(ax+by)=0, \qquad \text{Ans. } \frac{2}{3}\left(a^2+b^2\right)^{\frac{3}{2}}\pi.$$

4. Evaluate $\iint (1+x^2+y^2)^{-2}dx\,dy$,

taken over a triangle whose vertices are (0, 0). (2, 0) and $(1, \sqrt{3})$

Ans. $\frac{\sqrt{3}}{2} \tan^{-1}\frac{1}{2}$.

(C. H. 1962)

5. Evaluate

$$\iint \sqrt{\frac{36-9x^2-4y^2}{36+9x^2+4y^2}}\,dx\,dy$$

taken over the positive quadrant of the ellipse

$$\frac{x^2}{2^2}+\frac{y^2}{3^2}=1.$$ Ans. $\frac{9\pi}{2}(\pi-2)$

6. Evaluate $\iint (2a^2-2ax-2ay-x^2+y^2)dx\ dy$,

the region of integration being the circle

$$x^2+y^2+2ax+2ay-2a^2=0$$ (C. H. 1962, 63)

7. Evaluate $\iint (x^2+y^2)\ dxdy$ over the region enclosed by the triangle having the vertices (0, 0), (1, 0), (1, 1). (C. H. 1965)

8. Evaluate $\iiint z^2dxdydz$ extended over the hemisphere $z \geqslant 0$, $x^2+y^2+z^2 \leqslant a^2$. Ans. $\frac{2}{15}\pi a$. (C. H. 1964)

9. Prove that $\iiint \quad +nz)^2dxdydz=\frac{4}{15}\pi(l^2+m^2+n^2)$ taken throughout the sphere $x^2+y^2+z^2=1$.

10. Evaluate $\iiint \frac{dxdydz}{x^2+y^2+(z-2)^2}$ extended over the sphere $x^2+y^2+z^2 \leqslant 1$. Ans. $\pi(2-\frac{3}{2}\log 3)$,

PART II

APPLICATION OF ANALYSIS

CHAPTER VIII

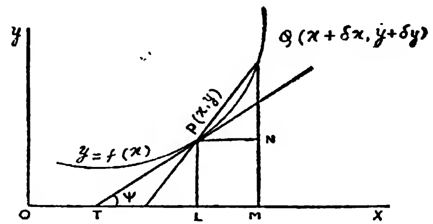
TANGENT AND NORMAL

8.1. Equation to the tangent to a curve in cartesian form.

Let $P(x, y)$ be a point on the curve $y=f(x)$ continuous at P and let $Q(x+\partial x, y+\partial y)$ be a point on the curve very near to P.

Then equation to the secant PQ is

$$Y-y=\frac{y+\partial y-y}{x+\partial x-x}(X-x)=\frac{\partial y}{\partial x}(X-x).$$



Now as $\partial x \to 0$, the point Q on the curve tends to the point P and the secant QP in the limit, becomes the tangent at P.

$\therefore$ The equation to the tangent at P is

$$Y-y=\underset{\partial x \to 0}{Lt}\ \frac{\partial y}{\partial x}(X-x)$$

$$=\frac{dy}{dx}(X-x), \text{ provided the limit exists and finite.}$$

Hence, tangent to the curve $y=f(x)$ at (x, y), not parallel to y axis, is

$$Y-y=\frac{dy}{dx}(X-x). \qquad \cdots \qquad \cdots \qquad (1)$$

If the equation to the curve be $f(x, y)=0$

Then $\quad \dfrac{dy}{dx}=-\dfrac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}$

$\therefore$ The equation to the tangent at any point (x, y) on the curve $f(x, y)=0$ becomes

$$Y-y=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}(X-x).$$

or, $$(X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}=0 \quad \cdots \quad \cdots \quad (2)$$

Again, if $f(x, y)$ be a rational algebraic function of x, y of degree n, then by multiplying each of its term by a suitable power of z, we can make $f(x, y)$ a homogeneous function of x, y, z. Then by Euler's Theorem

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf(x, y, z)=0$$

or, $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=-z\frac{\partial f}{\partial z}$$

So the equation (2) takes the form

$$X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}=x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$$
$$=-z\frac{\partial f}{\partial z}.$$

Hence, for the sake of symmetry, changing z by Z, the equation to the tangent becomes

$$X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}+Z\frac{\partial f}{\partial z}=0. \quad \cdots \quad \cdots \quad (3)$$

where the symbols Z and z are to be put equal to unity after differentiation.

In case, the curve be of the parametric form $x=f(t)$, $y=\phi(t)$, then

$$\frac{dy}{dx}=\frac{\phi'(t)}{f'(t)}.$$

So that equation to the tangent at the point t is, from (1),

$$\{X-f(t)\}\phi'(t)-\{Y-\phi(t)\}f'(t)=0. \quad \cdots \quad (4)$$

8.2. Equation to the Normal to a curve in cartesian form.

Equation to the tangent to the curve $y=f(x)$, not parallel to y axis, at any point (x, y) on the curve is

$Y-y=\frac{dy}{dx}(X-x)$ where $\frac{dy}{dx}$ denotes the slope of the tangent at (x, y).

So the equation to the normal at (x, y) whose slope is $-1\Big/\frac{dy}{dx}$, is

$$Y-y=-\frac{1}{\frac{dy}{dx}}(X-x)$$

$$\text{or,}\quad (X-x)+(Y-y)\frac{dy}{dx}=0. \qquad \cdots \quad (1)$$

If the curve be of the form $f(x, y)=0$, then since $\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$, the equation to the normal becomes

$$(X-x)\frac{\partial f}{\partial y}-(Y-y)\frac{\partial f}{\partial x}=0.$$

$$\text{or,}\quad \frac{X-x}{\frac{\partial f}{\partial x}}=\frac{Y-y}{\frac{\partial f}{\partial y}} \qquad \cdots \qquad \cdots \quad (2)$$

In case, the equation to the curve be of the parametric form $x=f(t)$, $y=\phi(t)$ the equation to the normal, from (1), takes the form

$$\{X-f(t)\}f'(t)+\{y-\phi(t)\}\phi'(t)=0.$$

Example :

Ex. 1. Find the equation to the tangent at any point (x, y) of the curve $\frac{x^p}{a^p}+\frac{y^p}{b^p}=1$

Let $\quad f(x, y)=\frac{x^p}{a^p}+\frac{y^p}{b^p}-1=0$

Then $\dfrac{\partial f}{\partial x}=\dfrac{px^{p-1}}{a^p}$; $\dfrac{\partial f}{\partial y}=\dfrac{py^{p-1}}{b^p}$

$$\therefore \quad \frac{dy}{dx}=-\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}=-\frac{b^p x^{p-1}}{a^p y^{p-1}}.$$

So, the equation to the tangent at (x, y) is

$$Y-y=-\frac{b^p x^{p-1}}{a^p y^{p-1}}(X-x)$$

$$\text{or,}\quad \frac{Xx^{p-1}}{a^p}+\frac{Yy^{p-1}}{b^p}=\frac{x^p}{a^p}+\frac{y^p}{b^p}=1$$

Alternative :

Let $f(x, y, z)=\dfrac{x^p}{a^p}+\dfrac{y^p}{b^p}-z^p$

$$\frac{\partial f}{\partial x}=\frac{px^{p-1}}{a^p},\quad \frac{\partial f}{\partial y}=\frac{py^{p-1}}{b^p},\quad \frac{\partial f}{\partial z}=-pz^{p-1}$$

So the equation of the tangent is

$$X\frac{px^{p-1}}{a^p}+Y\frac{py^{p-1}}{b^p}-Zpz^{p-1}=0 \text{ [See (3) Art. 8.1]}$$

Putting $Z=z=1$, we now get

$\dfrac{Xx^{p-1}}{a^p}+\dfrac{Yy^{p-1}}{b^p}=1$ as the required equation of the tangent at (x, y).

Ex. 2. Show that the straight line $lx+my=n$ touches the curve $\dfrac{x^p}{a^p}+\dfrac{y^p}{b^p}=1$

if $(al)^{p/(p-1)}+(bm)^{p/(p-1)}=(n)^{p/(p-1)}$, $p\neq 1$. (C. H. 1969)

The equation of the tangent to the curve at any point (x, y) is

$\dfrac{Xx^{p-1}}{a^p}+\dfrac{Yy^{p-1}}{b^p}=1$ where X, Y are the current co-ordinates.

By the given condition, co-ordinates is the same as

$lx+my=n$ where x, y are the current co-ordinates.

Equating co-efficients of the current co-ordinates

$$\frac{la^p}{x^{p-1}}=\frac{mb^p}{y^{p-1}}=n$$

$$\therefore \quad la=\frac{nx^{p-1}}{a^{p-1}} \text{ and } mb=\frac{ny^{p-1}}{b^{p-1}}$$

So that

$$(al)^{p/(p-1)}+(bm)^{p/(p-1)}=n^{p/(p-1)}\left\{\frac{x^p}{a^p}+\frac{y^p}{b^p}\right\}=n^{p/(p-1)}$$

Ex. 3. Prove that all points of the curve

$y^2=4a\left\{x+a \sin \frac{x}{a}\right\}$ at which the tangent is parallel to the axis of x lie on a parabola. (C. H. 1968)

$$\because \quad y^2=4a\left\{x+a \sin \frac{x}{a}\right\}$$

$$\therefore \quad 2y\frac{dy}{dx}=4a\left\{1+\cos \frac{x}{a}\right\}. \quad \cdots \quad \cdots \quad (1)$$

If the tangent is parallel to x axis, then $\frac{dy}{dx}=0$

$$\therefore \quad \text{From (1)} \quad 1+\cos \frac{x}{a}=0$$

$$\text{or,} \quad \cos \frac{x}{a}=-1=\cos \pi$$

$$\therefore \quad \frac{x}{a}=2n\pi \pm \pi$$

For this value of x, the tangent is parallel to x axis and the given equation reduces to

$$y^2=4a\{x+a \sin (2n\pi \pm \pi)\}$$
$$=4ax, \text{ which is a parabola.}$$

Hence, the required locus is a parabola.

Ex. 4. Show that equation to the normal to the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ may be expressed in the form

$$\mathbf{x \sin \phi - y \cos \phi + a \cos 2\phi = 0.}$$

From the given equation, $\dfrac{\partial f}{\partial x}=\dfrac{2}{3}x^{-\frac{1}{3}}$, $\dfrac{\partial f}{\partial y}=\dfrac{2}{3}y^{-\frac{1}{3}}$

$$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}=-\left(\frac{y}{x}\right)^{-\frac{1}{3}}.$$

$\Rightarrow$Slope of the normal at any point $(x, y)=\left(\dfrac{x}{y}\right)^{\frac{1}{3}}$. But from the form of the normal given, its slope $=\tan\phi$

$\therefore \quad \tan\phi = x^{1/3}/y^{1/3}$

or, $\quad x^{1/3}=y^{1/3}\tan\phi$

So, if we now solve for x and y the equations

$$x^{2/3}+y^{2/3}=a^{2/3} \quad \cdots \quad \cdots \quad (1)$$

and $\quad x^{1/3}-y^{1/3}\tan\phi=0 \quad \cdots \quad \cdots \quad (2)$

we shall get the co-ordinates of the point on the curve at which the slope is $\tan\phi$.

To solve (1) and (2)

$$y^{2/3}\tan^2\phi+y^{2/3}=a^{2/3}$$

or, $\quad y^{2/3}\sec^2\phi=a^{2/3}$

$\therefore \quad y=a\cos^3\phi.$

So from (2) $x=a\sin^3\phi$.

Hence, the equation to the normal to the given curve at $(a\sin^3\phi,\ a\cos^3\phi)$ is

$$y-a\cos^3\phi=\tan\phi\,(x-a\sin^3\phi)$$

or, $\quad y\cos\phi-a\cos^4\phi=x\sin\phi-a\sin^4\phi$

or, $\quad x\sin\phi-y\cos\phi=a(\sin^4\phi-\cos^4\phi).$

$$=a\,(\sin^2\phi-\cos^2\phi)$$

$$=-a\cos 2\phi.$$

$\Rightarrow x\sin\phi-y\cos\phi+a\cos 2\phi=0$ as the equation of the normal.

8.3. Equation of the Tangent at the origin.

If the equation to any curve passing through the origin be given (by a rational integral algebraic equation) in the form

$$(a_1x+b_1y)+(a_2x^2+b_2xy+c_2y^2)+\cdots+$$
$$\cdots+(a_nx^n+\cdots+l_ny^n)=0 \quad \cdots \quad (1)$$

then the equation to the tangent at the origin is obtained by equating to zero, the terms of the lowest degree in the equation.

Let $P(x, y)$ be any point on the curve very near to the origin 0.

Then equation to the line $0P$ is

$$Y=\frac{y}{x}X \quad \cdots \quad \cdots \quad \cdots \quad (2)$$

If the point P tends to the origin 0 i.e., when $x \to 0$, $y \to 0$, then the equation (2) becomes a tangent at the origin.

So, the equation to the tangent at 0,

is $$Y=\underset{\substack{x\to 0\\ y\to 0}}{Lt}\frac{y}{x}X, \text{ provided the limit exists and finite.} \quad \cdots \quad (3)$$

Let the tangent at 0 be not the y-axis. Then

$$\underset{\substack{x\to 0\\ y\to 0}}{Lt}\ \frac{y}{x} \text{ is finite}=m \text{ (say)}$$

Now dividing equation (1) by x

$$a_1+b_1\left(\frac{y}{x}\right)+a_2x+b_2y+c_2y\left(\frac{y}{x}\right)+\cdots\cdots=0$$

Proceeding to the limit as $x \to 0$, $y \to 0$, we get,

$$a_1+b_1m=0\cdots\cdots(4)$$

or, $m=-\dfrac{a_1}{b_1}$, provided $b_1 \neq 0$.

Thus if $b_1 \neq 0$, then equation to the tangent at the origin is from (3),

$$Y = mX = -\frac{a_1}{b_1}X$$

i.e., $a_1X + b_1Y = 0$.

i.e., $a_1x + b_1y = 0$, changing the current co-ordinates.

Again, if $b_1 = 0$, then from (4), $a_1 = 0$

$\therefore$ The equation (1) becomes

$$a_2x^2 + b_2xy + c_2y^2 + a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3 + \cdots = 0;$$

Dividing by x^2

$$a_2 + b_2\left(\frac{y}{x}\right) + c_2\left(\frac{y}{x}\right)^2 + a_3x + \cdots\cdots = 0.$$

Proceeding to the limit as $x \to 0$, $y \to 0$

$$a_2 + b_2m + c_2m^2 = 0 \cdots \cdot (5)$$

This gives the value of m, provided b_2 and c_2 are not both zero.

Now eliminating m between equation (5) and $y = mx$, the equation to the pair of tangent at the origin is

$$a_2X^2 + b_2XY + c_2Y^2 = 0$$

i.e., $a_2x^2 + b_2xy + c_2y^2 = 0$ changing the current co-ordinates.

Similarly, if $a_1 = b_1 = a_2 = b_2 = c_2 = 0$, then it can be shown that the equation to the tangent at the origin is the equation obtained by equating the 3rd degree terms of (1).

Hence, we see that, in general, the equation of the tangent or tangents at the origin is obtained by equating to zero, the terms of the lowest degree in the equation of the curve.

Note: If the tangent be the y axis, then also it can be shown that the theorem holds true.

For, in this case $\underset{\substack{x\to 0\\ y\to 0}}{Lt} \dfrac{x}{y}$ = tan of the angle made by the tangent with the y-axis = tan 0 = 0.

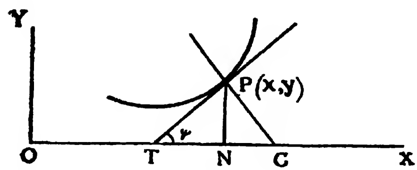
So dividing the equation (1) by y

and proceeding to the limit as $x\to 0$, $y\to 0$ the result follows.

8.4. Cartesian subtangent, subnormal, length of the tangent and length of the normal.

Let the tangent and normal at $P(x, y)$ on the curve $y=f(x)$ meet the x-axis at T and G so that $\angle PTX=\psi$. From P PN is drawn perpendicular to the x-axis.

Then TN is called the subtangent and NG is called the subnormal of the point P.



From the right angled triangle PTN

$$\tan\psi=\frac{PN}{TN}.$$

$$\text{or,}\quad \frac{dy}{dx}=\frac{y}{TN}$$

$$\text{or,}\quad TN=\frac{y}{\frac{dy}{dx}}=\frac{y}{y_1}.$$

Also, from the right angled triangle PNG

$$\tan\ GPN=\frac{NG}{PN}.$$

$$\text{or,}\quad \tan\psi=\frac{NG}{y}$$

$$\text{or,}\quad NG=y\frac{dy}{dx}=yy_1$$

Hence, the subtangent of the point $P(x, y)$ is y/y_1 and the subnormal of the point $P(x, y)$ is yy_1.

Again, length of the tangent $=PT=PN \operatorname{cosec} \psi$

$$=y\sqrt{1+\cot^2\psi}=y\sqrt{1+\frac{1}{{y_1}^2}}$$

$$=\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}.$$

Length of the normal $=PG=y \sec \psi=y\sqrt{1+\tan^2\psi}$

$$=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

8.5. Angle of intersection of two curves.

Let the two curves $f(x, y)=0$ and $\phi(x, y)=0$ intersect at (x, y).

The equation to the tangents to the two curves at (x, y) are

$$X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}-\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)=0$$

$$\text{and } X\frac{\partial \phi}{\partial x}+Y\frac{\partial \phi}{\partial y}-\left(x\frac{\partial \phi}{\partial x}+y\frac{\partial \phi}{\partial y}\right)=0.$$

Now the angle between two curves is the angle between these two tangents at (x, y).

Hence, if α be this angle, then

$$\tan \alpha=\frac{\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}\sim\frac{\partial \phi}{\partial x}\cdot\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial y}}.$$

Case I. If the two curves touch each other at (x, y) then $\alpha=0$ $\quad\therefore\ \tan \alpha=0.$

$$\Rightarrow\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}\sim\frac{\partial \phi}{\partial x}\cdot\frac{\partial f}{\partial y}=0.$$

$$\Rightarrow f_x\phi_y=\phi_x f_y$$

$$\Rightarrow f_x/\phi_x=f_y/\phi_y.$$

Case II. If the two curves cut orthogonally at (x, y), then

$$\tan \alpha = \pi/2 \qquad \therefore \cot \alpha = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial \phi}{\partial y} = 0$$

$$\Rightarrow f_x \phi_x + f_y \phi_y = 0.$$

Theorem :

Let $\xi = \phi(x, y)$, $\eta = \psi(x, y)$ where ϕ and ψ are differentiable functions of x and y. Show that if $\phi_x - \psi_y = 0$ and $\phi_y + \psi_x = 0$, then the angle between the curves $F(x, y) = 0$, $G(x, y) = 0$ in the xy-plane is equal to the angle between the curves $\phi(\xi, \eta) = 0$, $\psi(\xi, \eta) = 0$ in the $\xi\eta$-plane, where $\phi(\xi, \eta) = F(x, y)$, $\psi(\xi, \eta) = G(x, y)$, F, G, ϕ, ψ all being differentiable functions of the respective variables.

(C. H. 1967)

Let θ be the angle between $F(x, y) = 0$ and $G(x, y) = 0$.

$$\text{Then } \tan \theta = \frac{\dfrac{\partial F}{\partial x} \cdot \dfrac{\partial G}{\partial y} - \dfrac{\partial G}{\partial x} \cdot \dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial x} \cdot \dfrac{\partial G}{\partial x} + \dfrac{\partial F}{\partial y} \cdot \dfrac{\partial G}{\partial y}}. \quad \cdots \quad \cdots \quad (1)$$

If θ' be the angle between $\phi(\xi, \eta) = 0$, $\psi(\xi, \eta) = 0$

$$\text{Then } \tan \theta' = \frac{\dfrac{\partial \phi}{\partial \xi} \dfrac{\partial \psi}{\partial \eta} - \dfrac{\partial \psi}{\partial \xi} \dfrac{\partial \phi}{\partial \eta}}{\dfrac{\partial \phi}{\partial \xi} \dfrac{\partial \psi}{\partial \xi} + \dfrac{\partial \phi}{\partial \eta} \dfrac{\partial \psi}{\partial \eta}} \quad \cdots \quad \cdots \quad (2)$$

We are to prove $\theta = \theta'$.

From $F(x, y) = \phi(\xi, \eta)$

$$\frac{\partial F}{\partial x} = \frac{\partial \phi}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \phi}{\partial \eta} \frac{\partial \eta}{\partial x}$$

[$\because$ $\xi = \phi(x, y)$

$\eta = \psi(x, y)$

and $\phi_x = \psi_y$]

$$=\frac{\partial\phi}{\partial\xi}\,\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial\eta}\,\frac{\partial\psi}{\partial x}$$

$$=\frac{\partial\phi}{\partial\xi}\cdot\frac{\partial\psi}{\partial y}+\frac{\partial\phi}{\partial\eta}\cdot\frac{\partial\psi}{\partial x}$$

$$=\phi_\xi\psi_y+\phi_\eta\psi_x.$$

$$\frac{\partial F}{\partial y}=\frac{\partial\phi}{\partial\xi}\,\frac{\partial\xi}{\partial y}+\frac{\partial\phi}{\partial\eta}\,\frac{\partial\eta}{\partial y}$$

$$=\frac{\partial\phi}{\partial\xi}\,\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial\eta}\,\frac{\partial\psi}{\partial y}$$

$$[\because\quad \phi_y=-\psi_x]$$

$$=-\frac{\partial\phi}{\partial\xi}\cdot\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial\eta}\cdot\frac{\partial\psi}{\partial y}$$

$$=\phi_\eta\psi_y\quad\phi_\xi\psi_x.$$

From $\quad G(x,\,y)=\psi(\xi,\,\eta)$

$$\frac{\partial G}{\partial x}=\frac{\partial\psi}{\partial\xi}\,\frac{\partial\xi}{\partial x}+\frac{\partial\psi}{\partial\eta}\,\frac{\partial\eta}{\partial x}$$

$$=\frac{\partial\psi}{\partial\xi}\,\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial\eta}\,\frac{\partial\psi}{\partial x}$$

$$=\frac{\partial\phi}{\partial x}\,\frac{\partial\psi}{\partial\xi}\quad\frac{\partial\phi}{\partial y}\,\frac{\partial\psi}{\partial\eta}$$

$$=\phi_x\psi_\xi-\phi_y\psi_\eta.$$

$$\frac{\partial G}{\partial y}=\frac{\partial\psi}{\partial\xi}\,\frac{\partial\xi}{\partial y}+\frac{\partial\psi}{\partial\eta}\,\frac{\partial\eta}{\partial y}$$

$$=\frac{\partial\psi}{\partial\xi}\,\frac{\partial\phi}{\partial y}+\frac{\partial\psi}{\partial\eta}\,\frac{\partial\psi}{\partial y}$$

$$=\frac{\partial\phi}{\partial y}\,\frac{\partial\psi}{\partial\xi}+\frac{\partial\phi}{\partial x}\,\frac{\partial\psi}{\partial\eta}$$

$$=\phi_y\psi_\xi+\phi_x\psi_\eta$$

So putting in (1), we get $\tan\theta$

$$=\frac{(\phi_\xi\psi_y+\phi_\eta\psi_x)(\phi_y\psi_\xi+\phi_x\psi_\eta)-(\phi_x\psi_\xi-\phi_y\psi_\eta)(\phi_\eta\psi_y-\phi_\xi\psi_x)}{(\phi_\xi\psi_y+\phi_\eta\psi_x)(\phi_x\psi_\xi-\phi_y\psi_\eta)+(\phi_\eta\psi_y-\phi_\xi\psi_x)(\phi_y\psi_\xi+\phi_x\psi_\eta)}\qquad\ldots\qquad\ldots\qquad(3)$$

Numerator

$$=\phi_\xi\psi_y\phi_y\psi_\xi+\phi_\xi\psi_y\phi_x\psi_\eta+\phi_\eta\psi_x\phi_y\psi_\xi+\phi_\eta\psi_x\phi_x\psi_\eta$$

$$-\phi_x\psi_\xi\phi_\eta\psi_y+\phi_x\psi_\xi\phi_\xi\psi_x+\phi_y\psi_\eta\phi_\eta\psi_y-\phi_y\psi_\eta\phi_\xi\psi_x$$

$$=\phi_\xi\psi_\eta(\phi_x\psi_y-\psi_x\phi_y)-\psi_\xi\phi_\eta(\phi_x\psi_y-\psi_x\phi_y)$$

$$+\phi_\xi\psi_\xi(\phi_y\psi_y+\phi_x\psi_x)+\phi_\eta\psi_\eta(\phi_x\psi_x+\phi_y\psi_y)$$

$$[\because\quad \phi_x=\psi_y \text{ and } \phi_y=-\psi_x \quad\therefore\quad \phi_x\psi_x+\phi_y\psi_y=0\ ]$$

$$=(\phi_\xi\psi_\eta-\psi_\xi\phi_\eta)\,(\phi_x\psi_y-\psi_x\phi_y)$$

Denominator

$$=\phi_\xi\psi_y\phi_x\psi_\xi-\phi_\xi\psi_y\phi_y\psi_\eta+\phi_\eta\psi_x\phi_x\psi_\xi-\phi_\eta\psi_x\phi_y\psi_\eta$$

$$+\phi_\eta\psi_y\phi_y\psi_\xi+\phi_\eta\psi_y\phi_x\psi_\eta-\phi_\xi\psi_x\phi_y\psi_\xi-\phi_\xi\psi_x\phi_x\psi_\eta$$

$$=\phi_\xi\psi_\xi\,(\phi_x\psi_y-\psi_x\phi_y)+\phi_\eta\psi_\eta\,(\phi_x\psi_y-\phi_y\psi_x)$$

$$-\phi_\xi\psi_\eta(\phi_y\psi_y+\phi_x\psi_x)+\phi_\eta\psi_\xi\,(\phi_x\psi_x+\phi_y\psi_y)$$

$$=(\phi_\xi\psi_\xi+\phi_\eta\psi_\eta)\,(\phi_x\psi_y-\psi_x\phi_y)$$

So from (3)

$$\tan\theta=\frac{\phi_\xi\psi_\eta-\psi_\xi\phi_\eta}{\phi_\xi\psi_\xi+\phi_\eta\psi_\eta}$$

$$=\frac{\dfrac{\partial\phi}{\partial\xi}\cdot\dfrac{\partial\psi}{\partial\eta}-\dfrac{\partial\psi}{\partial\xi}\,\dfrac{\partial\phi}{\partial\eta}}{\dfrac{\partial\phi}{\partial\xi}\,\dfrac{\partial\psi}{\partial\xi}+\dfrac{\partial\phi}{\partial\eta}\,\dfrac{\partial\psi}{\partial\eta}}$$

$=\tan\theta'$ by (2)

Hence, $\theta=\theta'$.

Theorem :

Let $\xi=\phi(x, y)$ and $\eta=\psi(x, y)$ where ϕ and ψ are differentiable functions. If the angle between the curves $F(x, y)=0$,

G(x, y)=0 in the xy-plane is equal in magnitude to the angle between the curves $\phi(\xi, \eta)=0$, $\Gamma(\xi, \eta)=0$ in the $\xi\eta$-plane where

$$F(x, y)=\phi(\xi, \eta) \text{ and } G(x, y)=\Gamma(\xi, \eta)$$

according to the transformation in question, F, G, ϕ, Γ all being differentiable functions of the respective variables, then show that

$$\phi_x - \psi_y = 0,\ \phi_y + \psi_x = 0.$$

$$\text{or,}\quad \phi_x + \psi_y = 0,\ \phi_y - \psi_x = 0. \qquad \textbf{(C. H. 1970)}$$

From $F(x, y)=\phi(\xi, \eta)$

$$\frac{\partial F}{\partial x}=\frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial x}=\frac{\partial \phi}{d\xi}\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial \eta}\frac{\partial \psi}{\partial x} \qquad \left[\because \begin{array}{l}\xi=\phi(x, y)\\ \eta=\psi(x, y)\end{array}\right]$$

$$=\phi_\xi \phi_x + \phi_\eta \psi_x$$

$$\frac{\partial F}{\partial y}=\frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial y}=\frac{\partial \phi}{\partial \xi}\frac{\partial \phi}{\partial y}+\frac{\partial \phi}{\partial \eta}\frac{\partial \psi}{\partial y}$$

$$=\phi_\xi \phi_y + \phi_\eta \psi_y.$$

From $G(x, y)=\Gamma(\xi, \eta)$

$$\frac{\partial G}{\partial x}=\frac{\partial \Gamma}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial \Gamma}{\partial \eta}\frac{\partial \eta}{\partial x}=\frac{\partial \Gamma}{\partial \xi}\frac{\partial \phi}{\partial x}+\frac{\partial \Gamma}{\partial \eta}\frac{\partial \psi}{\partial x}$$

$$=\Gamma_\xi \phi_x + \Gamma_\eta \psi_x$$

$$\frac{\partial G}{\partial y}=\frac{\partial \Gamma}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial \Gamma}{\partial \eta}\frac{\partial \eta}{\partial y}=\frac{\partial \Gamma}{\partial \xi}\frac{\partial \phi}{\partial y}+\frac{\partial \Gamma}{\partial \eta}\frac{\partial \psi}{\partial y}$$

$$=\Gamma_\xi \phi_y + \Gamma_\eta \psi_y.$$

Now if θ be the angle between the curves $F(x, y)=0$, and $G(x, y)=0$, then

$$\tan\theta = \pm\frac{F_x G_y - G_x F_y}{F_x G_x + F_y G_y}$$

$$=\pm\frac{(\phi_\xi\phi_x+\phi_\eta\psi_x)(\Gamma_\xi\phi_y+\Gamma_\eta\psi_y)-(\Gamma_\xi\phi_x+\Gamma_\eta\psi_x)(\phi_\xi\phi_y+\phi_\eta\psi_y)}{(\phi_\xi\phi_x+\phi_\eta\psi_x)(\Gamma_\xi\phi_x+\Gamma_\eta\psi_x)+(\phi_\xi\phi_y+\phi_\eta\psi_y)(\Gamma_\xi\phi_y+\Gamma_\eta\psi_y)}$$

$$=\frac{\Gamma_\xi\phi_\eta(\psi_x\phi_y-\phi_x\psi_y)}{\Gamma_\xi\phi_\xi(\phi_x^2+\phi_y^2)+\Gamma_\eta\phi_\eta(\psi_x^2+\psi_y^2)+\Gamma_\eta\phi_\xi(\phi_x\psi_x+\phi_y\psi_y)+\Gamma_\eta\phi_\xi(\phi_x\psi_y-\psi_x\phi_y)+\Gamma_\xi\phi_\eta(\phi_x\psi_x+\psi_y\phi_y)}$$

$$=\frac{(\psi_x\phi_y-\phi_x\psi_y)(\Gamma_\xi\phi_\eta-\Gamma_\eta\phi_\xi)}{\Gamma_\xi\phi_\xi(\phi_x^2+\phi_y^2)+\Gamma_\eta\phi_\eta(\psi_x^2+\psi_y^2)+(\Gamma_\eta\phi_\xi+\Gamma_\xi\psi_x)\times(\phi_x\psi_x+\phi_y\psi_y)} \quad \cdots \quad (1)$$

Again, if θ' be the angle between the curves $\phi(\xi, \eta)=0$, $\Gamma(\xi, \eta)=0$, then

$$\tan\theta'=\pm\frac{\phi_\xi\Gamma_\eta-\Gamma_\xi\phi_\eta}{\phi_\xi\Gamma_\xi+\phi_\eta\Gamma_\eta} \quad \cdots \quad \cdots \quad (2)$$

If $\theta=\theta'$, then $\tan\theta=\tan\theta'$

From (1) and (2)

$$\frac{(\psi_x\phi_y-\phi_x\psi_y)(\Gamma_\xi\phi_\eta-\Gamma_\eta\phi_\xi)}{\Gamma_\xi\phi_\xi(\phi_x^2+\phi_y^2)+\Gamma_\eta\phi_\eta(\psi_x^2+\psi_y^2)+(\Gamma_\eta\phi_\xi+\Gamma_\xi\psi_x)(\phi_x\psi_x+\phi_y\psi_y)}$$

$$=\pm\frac{\phi_\xi\Gamma_\eta-\Gamma_\xi\phi_\eta}{\phi_\xi\Gamma_\xi+\phi_\eta\Gamma_\eta}$$

or, $$\pm(\phi_x\psi_y-\psi_x\phi_y)(\phi_\xi\Gamma_\xi+\phi_\eta\Gamma_\eta)$$
$$=\Gamma_\xi\phi_\xi(\phi_x^2+\phi_y^2)+\Gamma_\eta\phi_\eta(\psi_x^2+\psi_y^2)$$
$$+(\Gamma_\eta\phi_\xi+\Gamma_\xi\psi_x)(\phi_x\psi_x+\phi_y\psi_y).$$

This is identically satisfied if

$$\phi_x=\psi_y, \qquad \phi_y=-\psi_x$$

or, $$\phi_x=-\psi_y, \quad \phi_y=\psi_x$$

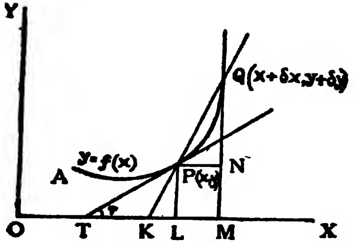
according as we take the upper sign or the lower sign.

Hence $\phi_x-\psi_y=0, \qquad \phi_y+\psi_x=0$

or, $\phi_x+\psi_y=0, \qquad \phi_y-\psi_x=0.$

8.6. Values of sin ψ, cos ψ and tan ψ.

Let $P(x, y)$ be a point on the curve $y=f(x)$ at a curved distance s from some fixed point A on the curve. Let $Q(x+\delta x, y+\delta y)$ be a point on the curve very near to P.



Let the join of QP be produced to meet the x axis at K and the tangent at P to meet at T on the x axis. Draw PL, QM perpendicular to x axis and PN perpendicular to QM.

Then $\sin QPN = \dfrac{QN}{QP} = \dfrac{\delta y}{\delta s} \cdot \dfrac{\delta s}{QP}$.

or, $\underset{Q \to P}{Lt} \sin QPN = \underset{Q \to P}{Lt} \dfrac{\delta y}{\delta s} \cdot \underset{Q \to P}{Lt} \dfrac{\delta s}{QP}$.

or, $\underset{Q \to P}{Lt} \sin PKL = \underset{\delta s \to 0}{Lt} \dfrac{\delta y}{\delta s} \quad \underset{Q \to P}{Lt} \dfrac{\text{arc } PQ}{\text{chord } PQ}$

$$= \frac{dy}{ds} \cdot 1. \tag{1}$$

[$\because$ as $Q \to P$, $\delta s \to 0$]

But as $Q \to P$ the chord QP becomes a tangent at P and $\angle PKL \to \psi$.

Hence, from (1) $\sin \psi = \dfrac{dy}{ds}$.

Also $\cos QPN = \dfrac{PN}{PQ} = \dfrac{\delta x}{\delta s} \cdot \dfrac{\delta s}{PQ}$

or, $\underset{Q \to P}{Lt} \cos QPN = \underset{Q \to P}{Lt} \dfrac{\delta x}{\delta s} \cdot \underset{Q \to P}{Lt} \dfrac{\delta s}{PQ}$

or, $\underset{Q \to P}{Lt} \cos PKL = \underset{\delta s \to 0}{Lt} \dfrac{\delta x}{\delta s} \cdot \underset{Q \to P}{Lt} \dfrac{\text{arc } PQ}{\text{chord } PQ}$

or, $\cos \psi = \dfrac{dx}{ds}$. [$\because$ as $Q \to P$ $\angle PKL \to \psi$ and $\delta s \to 0$]

Again, $\tan QPN = \frac{QN}{PN} = \frac{\delta y}{\delta x}$

or, $\underset{Q\to P}{Lt} \tan QPN = \underset{Q\to P}{Lt} \frac{\delta y}{\delta x}$

or, $\underset{Q\to P}{Lt} \tan PKL = \underset{\delta x\to 0}{Lt} \frac{\delta y}{\delta x}$ [$\because$ as $Q\to P$ $\delta x\to 0$ and $\angle PKL \to \psi$]

$\therefore \quad \tan \psi = \frac{dy}{dx}$.

8·7. Derivative of arc length.

In the last article, we have obtained the three important result

$$\sin \psi = \frac{dy}{ds} \quad \cdots \quad (1)$$

$$\cos \psi = \frac{dx}{ds} \quad \cdots \quad (2)$$

and $$\tan \psi = \frac{dy}{dx} \quad \cdots \quad (3)$$

So we write $$\frac{ds}{dx} = \sec \psi$$

$$= \sqrt{1+\tan^2\psi}$$

$$= \sqrt{1+\left(\frac{dy}{dx}\right)^2} \quad \cdots \quad (4)$$

$$\frac{ds}{dy} = \operatorname{cosec} \psi$$

$$= \sqrt{1+\cot^2\psi}$$

$$= \sqrt{1+\left(\frac{dx}{dy}\right)^2} \quad \cdots \quad (5)$$

The formula (4) and (5) can be written in the form

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

and $$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy.$$

Again, squaring (1), (2) and adding, we get

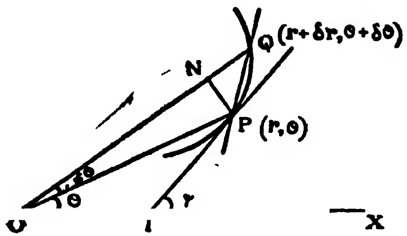
$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1.$$

8.8. Angle between the radius vector and the tangent at any point on the curve $r=f(\theta)$.

If ϕ be the angle between the tangent and the radius vector at any point in the curve $r=f(\theta)$, then

$$\tan\phi=r\frac{d\theta}{dr},\quad \sin\phi=r\frac{d\theta}{ds},\ \cos\phi=\frac{dr}{ds}.$$

Let $P(r, \theta)$ be a point on the curve $r=f(\theta)$ at an arc distance s from some fixed point on the curve, and let $Q(r+\delta r,\ \theta+\delta\theta)$ be a point on the curve very near to P at an arc distance $s+\delta s$, so that arc $PQ=\delta s$. Let ϕ be the angle between the tangent at P and the radius vector at P, so that $\angle OPT=\phi$.



Now as $Q\to P$, the value of $\delta\theta\to 0$, $\delta s\to 0$ and the secant QP tends to the tangent PT at P so that $\angle PQN\to\phi$.

From the right angled triangle PQN

$$\tan PQN=\frac{PN}{QN}=\frac{r\sin\delta\theta}{(r+\delta r)-r\cos\delta\theta}.$$

$$=\frac{r\sin\partial\theta}{r(1-\cos\partial\theta)+\partial r}$$

$$=\frac{r\left(\dfrac{\sin\delta\theta}{\delta\theta}\right)}{\dfrac{2r\sin^2\dfrac{\delta\theta}{2}}{\left(\dfrac{\partial\theta}{2}\right)^2}\cdot\dfrac{\delta\theta}{4}+\dfrac{\delta r}{\delta\theta}}.$$

$$\therefore\quad \underset{Q\to P}{Lt}\tan PQN=\frac{r\underset{Q\to P}{Lt}\left(\dfrac{\sin\delta\theta}{\delta\theta}\right)}{r\underset{Q\to P}{Lt}\left(\dfrac{\sin\dfrac{\delta\theta}{2}}{\dfrac{\delta\theta}{2}}\right)^2\cdot\dfrac{\delta\theta}{2}+\underset{Q\to P}{Lt}\dfrac{\delta r}{\delta\theta}}$$

$$\text{or,}\quad \tan\phi=\frac{r\underset{\delta\theta\to 0}{Lt}\left(\dfrac{\sin\delta\theta}{\delta\theta}\right)}{r\underset{\delta\theta\to 0}{Lt}\left(\dfrac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)^2\cdot\dfrac{\delta\theta}{2}+\underset{\delta\theta\to 0}{Lt}\dfrac{\delta r}{\delta\theta}}$$

$$=\frac{r.\,1}{r.\,1.\,0+\dfrac{dr}{d\theta}}=r\frac{d\theta}{dr}.$$

$$\text{Hence, } \tan\phi=r\frac{d\theta}{dr}\qquad\ldots\qquad\ldots\qquad(1)$$

$$\text{Again, } \sin PQN=\frac{PN}{PQ}$$

$$=\frac{r\sin\delta\theta}{\delta\theta}\cdot\frac{\delta\theta}{\delta s}\cdot\frac{\delta s}{PQ}$$

$$\text{or,}\quad \underset{Q\to P}{Lt}\sin PQN=r\underset{Q\to P}{Lt}\cdot\frac{\sin\delta\theta}{\delta\theta}\underset{Q\to P}{Lt}\frac{\delta\theta}{\delta s}\cdot\underset{Q\to P}{Lt}\frac{\delta s}{PQ}$$

$$\text{or,}\quad \sin\phi=r.\underset{\delta\theta\to 0}{Lt}\frac{\sin\delta\theta}{\delta\theta}\cdot\underset{\delta s\to 0}{Lt}\frac{\delta\theta}{\delta s}\underset{Q\to P}{Lt}\frac{\text{arc } PQ}{\text{chord } PQ}$$

$$=r.\,1.\,\frac{d\theta}{ds}.\,1$$

$$=r\frac{d\theta}{ds}$$

$$\text{Hence, } \sin\phi=r\frac{d\theta}{ds}\qquad\ldots\qquad\ldots\qquad(2)$$

$$\text{Also,}\quad \cos PQN=\frac{QN}{PQ}=\frac{(r+\delta r)-r\cos\delta\theta}{PQ}$$

$$=\frac{2r\sin^2\frac{\delta\theta}{2}+\delta r}{PQ}$$

$$=\frac{1}{2}r\delta\theta\left(\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)^2\cdot\frac{\delta\theta}{\delta s}\cdot\frac{\delta s}{PQ}+\frac{\delta r}{\delta s}\cdot\frac{\delta s}{PQ}$$

$$\text{or,}\quad \underset{Q\to P}{Lt} \cos PQN = \frac{1}{2} r \underset{Q\to P}{Lt}\ \theta \left(\frac{\sin \frac{\delta\theta}{2}}{\frac{\delta\theta}{2}} \right)^2$$

$$\times \frac{\delta\theta}{\delta s} \cdot \frac{\delta s}{PQ} + \underset{Q\to P}{Lt} \frac{\delta r}{\delta s} \cdot \frac{\delta s}{PQ}$$

$$= \frac{1}{2} r \underset{\delta\theta\to 0}{Lt} \partial\theta \left(\frac{\sin \frac{\delta\theta}{2}}{\frac{\delta\theta}{2}} \right)^2 \cdot \underset{\delta s\to 0}{Lt} \frac{\delta\theta}{\delta s} \cdot \underset{Q\to P}{Lt} \frac{\delta s}{PQ}$$

$$+ \underset{\delta s\to 0}{Lt} \frac{\delta r}{\delta s} \cdot \underset{Q\to P}{Lt} \frac{\delta s}{PQ}$$

$$\text{or,}\quad \cos \phi = \frac{1}{2} r \,.\, 0.\, 1 \,.\, \frac{d\theta}{ds} .\, 1 + \frac{dr}{ds} \,.\, 1$$

$$= \frac{dr}{ds}.$$

Hence, $\cos \phi = \dfrac{dr}{ds}$ .. ... (3)

Cor. 1. Squaring (2), (3) and adding we get

$$\left(\frac{dr}{ds}\right)^2 + r^2 \left(\frac{d\theta}{ds}\right)^2 = \cos^2\phi + \sin^2\phi = 1, \quad \cdots \quad (4)$$

Cor. 2. From (2) $\dfrac{1}{r}\dfrac{ds}{d\theta} = \operatorname{cosec} \phi$

$$= \sqrt{1+\cot^2\phi}$$

$$= \sqrt{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2} \quad \text{from (1)}$$

$$\therefore\quad \frac{ds}{d\theta} = \sqrt{r\delta + \left(\frac{dr}{d\theta}\right)^2} \quad \cdots \quad \cdots \quad (5)$$

Multiplying both sides of (5) by $\dfrac{d\theta}{dr}$

$$\frac{ds}{dr} = \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2} \quad \cdots \quad \cdots \quad (6)$$

Cor. 3. From (4), (5) and (6)

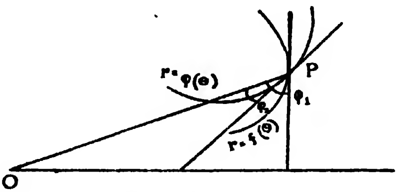
$$ds^2 = dr^2 + r^2 d\theta^2$$

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

and $$ds = \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}\, d\theta.$$

8.9. Angle of intersection between the two curves in polar co-ordinates.

Let the two curves $r = f(\theta)$ and $r = \phi(\theta)$ cut at P and let the tangent at P to the two curves make angles ϕ_1 and ϕ_2 with the radius vector OP.



Then if α be the angle between the two curves, then α is also the angle between the two tangent at P

$$\therefore \quad \alpha = \phi_1 - \phi_2$$

$$\therefore \quad \tan \alpha = \frac{\tan \phi_1 - \tan \phi_2}{1 + \tan \phi_1 \tan \phi_2} \tag{1}$$

But we have

$$\tan \phi_1 = r\frac{d\theta}{dr}$$

$$= \frac{r}{\dfrac{dr}{d\theta}}$$

$$= \frac{f(\theta)}{f'(\theta)}.$$

and $$\tan \phi_2 = \frac{r}{\dfrac{dr}{d\theta}} = \frac{\phi(\theta)}{\phi'(\theta)}$$

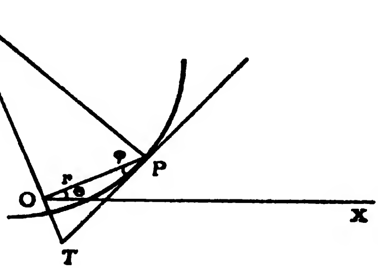
$\therefore$ From (1)

$$\tan\alpha=\frac{\dfrac{f(\theta)}{f'(\theta)}-\dfrac{\phi(\theta)}{\phi'(\theta)}}{1+\dfrac{f(\theta)}{f'(\theta)}\cdot\dfrac{\phi(\theta)}{\phi'(\theta)}}$$

$$=\frac{f(\theta)\phi'(\theta)-f'(\theta)\phi(\theta)}{f(\theta)\phi(\theta)+f'(\theta)\phi'(\theta)}.$$

8.10. Polar subtangent and polar subnormal.

Let $P(r, \theta)$ be any point on the curve $r=f(\theta)$ with respect to O as pole and OX as the fixed line. Let the tangent and normal at P meet the line through O perpendicular to the radius vector OP at T and G.



Then OT is the polar subtangent, and OG is the polar subnormal of the point P.

Let $\angle OPT=\phi$.

Then, $\tan\phi=r\dfrac{d\theta}{dr}$. ... ... (1)

But, from the right angled triangle OPT

$$\tan\phi=\frac{OT}{OP}=\frac{OT}{r}$$

$\therefore$ $OT=r\tan\phi=r^2\dfrac{d\theta}{dr}$ [from (1)]

and $\cot OPG=\dfrac{OP}{OG}=\dfrac{r}{OG}.$

or, $\quad \cot(90-\phi)=\frac{r}{OG}$.

or, $\quad OG=\frac{r}{\tan\phi}=\frac{r}{r\frac{d\theta}{dr}}=\frac{dr}{d\theta}$.

Hence, the polar subtangent $=r^2\frac{d\theta}{dr}$

and the polar subnormal $=\frac{dr}{d\theta}$

8.11. A relation between any point on the curve and the perpendicular distance from the pole on the tangent at the point.

Let p be the length of the perpendicular from the pole 0 on the tangent at $P(r, \theta)$ on the curve $r=f(\theta)$ with respect to O as pole and the tangent at P. Let the tangent at P and the radius vector OP make an angle ϕ.

Then $\quad p=r\sin\phi. \qquad \cdots \qquad \cdots \qquad (1)$

$$\therefore \quad \frac{1}{p^2}=\frac{1}{r^2}\operatorname{cosec}^2\phi=\frac{1}{r^2}(1+\cot^2\phi)$$

$$=\frac{1}{r^2}\left\{1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\}$$

$$=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2. \qquad \cdots \qquad (2)$$

If we write $u=\frac{1}{r}$

then $\quad \frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$.

so that $\quad \left(\frac{du}{d\theta}\right)^2=\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

$\therefore$ From (2) we get

$$\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2 \qquad \cdots \qquad (3)$$

8.12. Transformation from polar to cartesian form.

(*a*) To transform $\tan\phi = r\dfrac{d\theta}{dr}$

to the corresponding cartesian form.

Since $x = r\cos\theta,\ y = r\sin\theta$

$\therefore\ r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\dfrac{y}{x}$.

$$\therefore\ \frac{dr}{dx} = \frac{1}{2\sqrt{x^2+y^2}}\left(2x+2y\frac{dy}{dx}\right) = \frac{x+y\dfrac{dy}{dx}}{r}$$

$$\frac{d\theta}{dx} = \frac{1}{1+\dfrac{y^2}{x^2}}\cdot\frac{x\dfrac{dy}{dx}-y}{x^2} = \frac{x\dfrac{dy}{dx}-y}{r^2}.$$

Now, $$\frac{d\theta}{dr} = \frac{\dfrac{d\theta}{dx}}{\dfrac{dr}{dx}} = \frac{x\dfrac{dy}{dx}-y}{r^2}\cdot\frac{r}{x+y\dfrac{dy}{dx}}$$

$$\therefore\ r\frac{d\theta}{dr} = \frac{x\dfrac{dy}{dx}-y}{x+y\dfrac{dy}{dx}}. \qquad \cdots \qquad \cdots \qquad (1)$$

Hence, $$\tan\phi = \frac{x\dfrac{dy}{dx}-y}{x+y\dfrac{dy}{dx}}.$$

(*b*) To transform p = r sin ϕ to the corresponding cartesian form.

$$p = r\sin\phi = \frac{r}{\operatorname{cosec}\phi} = \frac{r}{\sqrt{1+\cot^2\phi}}$$

$$= \frac{r}{\sqrt{\left\{1+\dfrac{1}{r^2}\left(\dfrac{dr}{d\theta}\right)^2\right\}}} \qquad \cdots \qquad (1)$$

$\therefore$ From (1) of (*a*)

$$\frac{1}{r}\left(\frac{dr}{d\theta}\right)=\frac{x+y\frac{dy}{dx}}{x\frac{dy}{dx}-y}$$

$$\therefore \quad 1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2=1+\frac{\left(x+y\frac{dy}{dx}\right)^2}{\left(x\frac{dy}{dx}-y\right)^2}$$

$$=\frac{(x^2+y^2)\left(\frac{dy}{dx}\right)^2+(x^2+y^2)}{\left(x\frac{dy}{dx}-y\right)^2}$$

$$=\frac{r^2\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{\left(x\frac{dy}{dx}-y\right)^2}.$$

Hence, from (1)

$$p=\frac{x\frac{dy}{dx}-y}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}.$$

Illustrative Examples :

Ex. 1. In the curve $r^2=a^2\sin 2\theta$, prove that $\phi=2\theta$ where ϕ is the angle which the radius vector makes with the tangent. **(C. H. 1969)**

$$r^2=a^2 \sin 2\theta.$$

or, $\quad 2 \log r=2 \log a+\log \sin 2\theta.$

Differentiating w. r. to θ,

$$\frac{2}{r}\frac{dr}{d\theta}=\frac{1}{\sin 2\theta}\cdot 2\cos 2\theta.$$

or, $\quad \cot\phi=\cot 2\theta$

$\therefore \quad \phi=2\theta.$

Ex. 2. Find the angle of intersection of the curves $r^n = a^n \sec(n\theta + \alpha)$, $r^n = b^n \sec(n\theta + \beta)$. **(C. H. 1964)**

From the 1st curve

$n \log r = n \log a + \log \sec(n\theta + \alpha)$

$\therefore$ Differentiating w. r. to θ

$$n \frac{1}{r} \frac{dr}{d\theta} = \frac{1}{\sec(n\theta+\alpha)} n \sec(n\theta+\alpha) \tan(n\theta+\alpha)$$

or, $\frac{1}{r} \frac{dr}{d\theta} = \tan(n\theta+\alpha)$

or, $\cot \phi_1 = \tan(n\theta+\alpha)$

$$= \cot\left\{\frac{\pi}{2} - (n\theta+\alpha)\right\}$$

$\therefore \quad \phi_1 = \frac{\pi}{2} - (n\theta+\alpha)$

Similarly, from the 2nd curve

$\phi_2 = \frac{\pi}{2} - (n\theta+\beta)$

$\therefore$ Angle of intersection

$= (\phi_1 \sim \phi_2) = (\alpha \sim \beta)$

Ex. 3. Prove that the locus of the extremity of the polar subtangent of the curve $u + f(\theta) = 0$ is the curve $u = f'\left(\frac{\pi}{2} + \theta\right)$ where $u = \frac{1}{r}$.

$\because \quad u = -f(\theta)$, i.e., $\frac{1}{r} = -f(\theta)$

$\therefore \quad -\frac{1}{r^2} \frac{dr}{d\theta} = -f'(\theta)$ [See Fig. at page 168]

i.e., $r^2 \frac{d\theta}{dr} = \frac{1}{f'(\theta)}$

$$OT = r^2 \frac{d\theta}{dr} = \frac{1}{f'(\theta)} \quad \cdots \quad \cdots \quad (1)$$

Let (r', θ') be the polar coordinates of T, then from (1)

$$r' = \frac{1}{f'(\theta)} \quad \cdots \quad \cdots \quad \cdots \quad (2)$$

Also $\theta' = -\angle XOT = -(\angle POT - \angle POX)$

$$= \angle POX - \angle POT$$

$$= \theta - \frac{\pi}{2}$$

$$\therefore \quad \theta = \theta' + \frac{\pi}{2}.$$

So from (2) $r' = \dfrac{}{f'\left(\theta' + \frac{\pi}{2}\right)}.$

i.e., $u' = f'\left(\theta' + \frac{\pi}{2}\right).$

Hence, the locus of T is the curve

$$u = f'\left(\theta + \frac{\pi}{2}\right).$$

Ex. 4. Prove that the locus of the extremity of the polar subnormal of the curve $r = f(\theta)$ is

$$r = f'\left(\theta - \frac{\pi}{2}\right).$$

Hence, deduce that the locus of the extremity of the polar subnormal of the equiangular spiral $r = ae^{\theta \cot \alpha}$ is another equiangular spiral.

From the cuave $r = f(\theta)$

$$\frac{dr}{d\theta} = f'(\theta).$$

$$\therefore \quad OG = f'(\theta) \quad \left[\because \text{ the subnormal } OG = \frac{dr}{d\theta}.\right]$$

Let (r', θ') be the co-ordinate of G (See Fig. at page 168).

Then $r' = OG = f'(\theta)$. ... ... (1)

Again $\theta' = \angle XOG = \frac{\pi}{2} + \theta$.

$$\therefore \quad \theta = \theta' - \frac{\pi}{2}.$$

So from (1)

$$r' = f'\left(\theta' - \frac{\pi}{2}\right).$$

Hence, the locus of $G(r', \theta')$ is the curve

$$r = f'\left(\theta - \frac{\pi}{2}\right).$$

Deduction :

For the curve $r = ae^{\theta \cot \alpha}$

$$f'(\theta) = a \cot \alpha \, e^{\theta \cot \alpha} \qquad \because \quad r = f(\theta)$$

$\therefore$ Locus of the e xtremity G of the subnormal is

$$r = f'\left(\theta - \frac{\pi}{2}\right)$$

$$= a \cot \alpha \, e^{\left(\theta - \frac{\pi}{2}\right) \cot \alpha}$$

$$= a \cot \alpha \, e^{-\frac{\pi}{2} \cot \alpha} \, . \, e^{\theta \cot \alpha}$$

$$= Ae^{\theta \cot \alpha}$$

[Putting A for the constant portion $a \cot \alpha \, e^{-\frac{\pi}{2} \cot \alpha}$.]

Hence, the locus of 'G' is another equiangular spiral whose equation is

$$r = Ae^{\theta \cot \alpha}.$$

8.13. Pedal Equation to a curve.

A relation between p and r where p is the length of the perpendicular from the origin (or pole) on the tangent at any point of the curve and r is the distance of the point on the curve from the origin (or pole) is called the pedal equation to the curve.

How to find the pedal equation

(*a*) Let the equation of the curve be given in cartesian form

$$y = f(x) \quad \cdots \quad \cdots \quad (1)$$

Then, equation to the tangent at any point on (1) is

$$Y-y=\frac{dy}{dx}(X-x)$$

or, $$X\frac{dy}{dx}-Y+\left(y-x\frac{dy}{dx}\right)=0$$

So, the length of the perpendicular p from the origin on this tangent is

$$p=\frac{y-x\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}} \qquad (2)$$

Also $r^2=x^2+y^2$ ... ... (3)

Now eliminating x, y from (1), (2) and (3), a relation between p and r is obtained which gives the pedal equation to the curve.

(b) Let the equation to the curve be given in the polar form $r=f(\theta)$ ... ... (1)

Also, we have the relation

$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad \cdots \quad (2)$$

Eliminating θ between (1) and (2), we obtain the pedal equation, a relation between p and r.

But, instead of using (2), the pedal equation may conveniently be obtained by eliminating θ and ϕ between (1) and the following two relations

$$\tan\phi=r\frac{d\theta}{dr}$$

and $$p=r\sin\phi.$$

Illustrative Examples :

Ex. 1. Find the pedal equation of the curve whose polar equation $r^2=a^2\sin 2\theta$ is given. (C. H. 1969)

Here $2\log r=2\log a+\log\sin 2\theta$

or, $$\frac{2dr}{rd\theta}=\frac{2\cos 2\theta}{\sin 2\theta}$$

or, $$\cot\phi=\cot 2\theta$$

$$\therefore \quad \phi=2\theta.$$

Now $$p=r\sin\phi=r\sin 2\theta$$
$$=r\cdot\frac{r^2}{a}$$

or, $r^3=a^2p$ is the required pedal equation of the curve.

Ex. 2. Find the pedal equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with regard to the centre. **(C. H. 1966)**

Let $(a\cos\phi, b\sin\phi)$ be any point on the ellipse. Then, equation to the tangent at ϕ, is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi-1=0.$$

i.e., $$-\frac{x}{a}\cos\phi-\frac{y}{b}\sin\phi+1=0$$

So the length of the perpendicular from the origin on this tangent

$$p=\frac{1}{\sqrt{\frac{\cos^2\phi}{a^2}+\frac{\sin^2\phi}{b^2}}}=\frac{ab}{\sqrt{b^2\cos^2\phi+a^2\sin^2\phi}}.$$

$$=\frac{ab}{\sqrt{(a^2+b^2)-(a^2\cos^2\phi+b^2\sin^2\phi)}}$$

$$=\frac{ab}{\sqrt{a^2+b^2-r^2}}$$

$$[\because\ r^2=x^2+y^2=a^2\cos^2\phi+b^2\sin^2\phi.]$$

or, $$\frac{a^2b^2}{p^2}=a^2+b^2-r^2$$

is the required pedal equation of the ellipse with regard to its centre.

Ex. 3. Find the pedal equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with regard to one of its focus. (C. H. 1967)

Let $P(a\cos\phi, b\sin\phi)$ be any point on the ellipse and $S(ea, 0)$ be its focus on the positive side. Join SP and draw PN perpendicular to the major axis.

Then $SN=a\cos\phi-ea$ and $PN=b\sin\phi$

$$\therefore \quad r^2=SP^2=SN^2+PN^2=(a\cos\phi-ea)^2+b^2\sin^2\phi.$$
$$=a^2(\cos\phi-e)^2+a^2(1-e^2)\sin^2\phi$$
$$=a^2\{\cos^2\phi+e^2-2e\cos\phi+\sin^2\phi-e^2\sin^2\phi\}.$$
$$=a^2\{1+e^2-2e\cos\phi-e^2(1-\cos^2\phi)\}$$
$$=a^2\{1-2e\cos\phi+e^2\cos^2\phi\}$$
$$=a^2(1-e\cos\phi)^2. \quad \cdots \quad \cdots \quad (1)$$

Now equation to the tangent at ϕ is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi-1=0.$$

$\therefore$ Perpendicular from $S(ea, 0)$ is

$$p^2=\frac{(e\cos\phi-1)^2}{\frac{\cos^2\phi}{a^2}+\frac{\sin^2\phi}{b^2}}=\frac{a^2b^2(e\cos\phi-1)^2}{b^2\cos^2\phi+a^2\sin^2\phi}.$$

$$\text{or,}\quad \frac{b^2}{p^2}=\frac{b^2\cos^2\phi+a^2\sin^2\phi}{a^2(1-e\cos\phi)^2}$$
$$=\frac{a^2(1-e^2)\cos^2\phi+a^2\sin^2\phi}{a^2(1-e\cos\phi)^2}$$
$$=\frac{\cos^2\phi-e^2\cos^2\phi+\sin^2\phi}{(1-e\cos\phi)^2}$$
$$=\frac{1-e^2\cos^2\phi}{(1-e\cos\phi)^2}$$

$$\therefore \quad \frac{b^2}{p^2}+1=\frac{1-e^2\cos^2\phi+1-2e\cos\phi+e^2\cos^2\phi}{(1-e\cos\phi)^2}$$
$$=\frac{2(1-e\cos\phi)}{(1-e\cos\phi)^2}=\frac{2}{1-e\cos\phi}$$
$$=\frac{2}{r/a} \qquad \text{by (1)}$$

or, $\frac{b^2}{p^2}=\frac{2a}{r}-1$ is the required pedal equation of the ellipse with regard to one of its focus.

Exercise 8

1. Show that the portion of the normal to the curve $x=c(4\cos^3\theta-3\cos\theta)$ $y=c(4\sin^3\theta-9\sin\theta)$ intercepted between the axes is of constant length.

2. If the line $x\cos\alpha+y\sin\alpha=p$ touches the curve

$$\left(\frac{x}{a}\right)^{\frac{n}{n-1}}+\left(\frac{y}{b}\right)^{\frac{n}{n-1}}=1$$

prove that $(a\cos\alpha)^n+(b\sin\alpha)^n=p^n$. (C. H. 1973)

3. Show that the two curves

$ax^2+by^2=1$ and $cx^2+dy^2=1$

cut orthogonally if $\frac{1}{b}-\frac{1}{d}=\frac{1}{a}-\frac{1}{c}$.

4. Show that the two cardioides $r=a(1+\cos\theta)$, $r=b(1-\cos\theta)$ cut each other at right angles.

5. Show that the two curves $x^2+y^2=\sqrt{2}a^2$ and $x^2-y^2=a^2$ cut each other at an angle of $\pi/4$.

6. In the catenary $y=c\cosh\left(\frac{x}{c}\right)$ show that the length of the perpendicular from the foot of the ordinate on the tangent is of constant length and that the length of the normal at any point is y^2/c.

7. Show that in the curve $x^p y^q=a^{p+q}$

(a) the subtangent at any point varies as the abscissa of the point.

(b) the portion of the tangent intercepted between the axes is divided at its point of contact in a constant ratio.

8. Prove that the two curves $r^n=a^n\cos n\theta$ and $r^n=b^n\sin n\theta$ cut orthogonally.

9. Show that pedal equation of the parabola $y^2=4ax$ with regard to its vertex is

$$(p^2r^2+4a^2p^2)(4a^2+p^2)=(ar^2-ap^2)^2.$$

CHAPTER IX

ASYMPTOTES

9.1. Definition.

A straight line is said to be an Asymptote of a curve if, as a point P recedes to infinity along the curve away

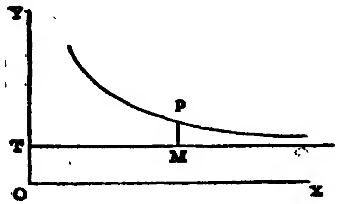


Fig. 1

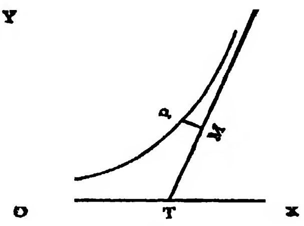


Fig. 2

from the origin, the perpendicular distance of P from the straight line tends to zero.

Theorem :

If $y = mx + c$ be an oblique asymptote then

$$\lim_{x\to\infty} (y - mx) = c$$

and $$\lim_{x\to\infty} \left(\frac{y}{x}\right) = m.$$

Let $y = mx + c$ be an oblique asymptote to the curve $y = f(x)$. Let p be the length of the perpendicular from any point $P(x, y)$ on the curve on the asymptote.

Then $$p = \frac{y - mx - c}{\sqrt{1+m^2}}.$$

$\because$ $y = mx + c$ is an asymptote to the curve

$$p = 0 \text{ as } x \to \infty$$

$$\Rightarrow \quad y - mx - c = 0 \text{ as } x \to \infty$$

$$\Rightarrow \quad \underset{x \to \infty}{Lt} (y - mx - c) = 0$$

i.e., $$\underset{x \to \infty}{Lt} (y - mx) = c.$$

Again, $$\frac{y}{x} - m = (y - mx)\frac{1}{x}.$$

$$\Rightarrow \quad \underset{x \to \infty}{Lt} \left\{ \left(\frac{y}{x}\right) - m \right\} = \underset{x \to \infty}{Lt} (y - mx) \cdot \underset{x \to \infty}{Lt} \frac{1}{x}$$

$$= c \cdot 0 = 0$$

$$\therefore \quad \underset{x \to \infty}{Lt} \left(\frac{y}{x}\right) = m$$

Ex. **Find the asymptotes to the curve**

$$\mathbf{y = xe^{1/x^2}}$$ *(C. H. 1969)*

Let $y = mx + c$ be an asymptote.

Then $$m = \underset{x \to \infty}{Lt} \left(\frac{y}{x}\right) = \underset{x \to \infty}{Lt} e^{1/x^2} = e^0 = 1.$$

and $$c = \underset{x \to \infty}{Lt} (y - mx)$$

$$= \underset{x \to \infty}{Lt} (y - x) \quad \because \quad m = 1$$

$$= \underset{x \to \infty}{Lt} \left(xe^{1/x^2} - x \right)$$

$$= \underset{x \to \infty}{Lt} \frac{e^{1/x^2} - 1}{\frac{1}{x}} \text{ form } \frac{0}{0}$$

$$= \underset{x \to \infty}{Lt} \frac{-\frac{2}{x^3} e^{1/x^2}}{-\frac{1}{x^2}}$$

$$= \underset{x \to \infty}{Lt} \frac{2e^{1/x^2}}{x} = \frac{2}{\infty} = 0.$$

So the required asymptote is $y = x$.

Theorem :

$y=mx+c$ is an asymptote of the curve $y=mx+c+f(x)$

where $\underset{x\to\infty}{Lt} f(x)=0$.

Proof : $y=mx+c+f(x)$

$$\Rightarrow \frac{y}{x}=m+\frac{c}{x}+\frac{1}{x}f(x)$$

$$\therefore \underset{x\to\infty}{Lt}\left(\frac{y}{x}\right)=m \cdots\cdots\cdots(1)$$

Again $y=mx+c+f(x)$

$$\Rightarrow y-mx=c+f(x)$$

$$\therefore \underset{x\to\infty}{Lt}(y-mx)=c+\underset{x\to\infty}{Lt}f(x)$$

$$=c \cdots\cdots\cdots(2)$$

From (1) and (2) it follows that

$$y=mx+c$$

is an asymptote.

Ex. Find the asymptotes of

$$y=\frac{x^3+4x-3}{x^2-4x+3}.$$

Since, $x^3+4x-3=(x+4)(x^2-4x+3)+(17x-15)$

$\therefore$ By actual division

$$y=(x+4)+\frac{17x-15}{x^2-4x+3} \cdots\cdots\cdots(1)$$

Now, since $\underset{x\to\infty}{Lt}\dfrac{17x-15}{x^2-4x+3}=\underset{x\to\infty}{Lt}\dfrac{17-\dfrac{15}{x}}{x\left(1-\dfrac{4}{x}+\dfrac{3}{x^2}\right)}=0$

From (1)

$y=x+4$ is an asymptote.

Again $y=\dfrac{x^2+4x-3}{(x-1)(x-3)}$

$\because$ $y\to\infty$ if $x=1$ and 3

$\therefore$ $x=1$, $x=3$ are also two other asymptotes. Hence, the three asymptotes are

$$x-1=0,\ x-3=0 \text{ and } x-y+4=0.$$

9.2. Position of a curve with respect to an asymptote.

Let the equation to the curve be

$$y=mx+c+f(x)\cdots\cdots(1)$$

where $\underset{x\to\infty}{Lt} f(x)=0.$

Then $y=mx+c$ is an asymptote to (1)$\cdots$(2)

Let y_1 and y_2 be the ordinates of the curve (1) and the asymptote (2) corresponding to any value of x.

i.e., $y_1=mx+c+f(x)$

and $y_2=mx+c$

$\therefore$ $y_1-y_2=f(x)$.

So it follows that

$y_1>y_2$ for those values of x for which $f(x)$ is positive.

$\therefore$ The curve lies above the asymptote for those values of x for which $f(x)$ is positive.

Again $y_1<y_2$ for those values of x for which $f(x)$ is negative.

$\therefore$ The curve lies below the asymptote for those values of x for which $f(x)$ is negative.

Ex. Find the asymptotes of

$$\mathbf{y=\frac{x^2-2x-1}{x}.}$$

and investigate the position of the curve and asymptotes.

We have, $y=(x-2)-\dfrac{1}{x}$

where $\underset{x \to \infty}{Lt} \frac{1}{x} = 0.$

$\therefore$ $y = x - 2$ is an asymptote.

Also $y \to \infty$ when $x = 0$.

$\therefore$ $x = 0$ is the other asymptote.

To investigate the position of the curve

$$y = x - 2 - \frac{1}{x} \text{ and the asymptote } y = x - 2.$$

Let y_1 and y_2 be the ordinates of the curve and the asymptote corresponding to the same abscissa.

$$\text{i. e.,} \quad y_1 = x - 2 - \frac{1}{x}$$

$$\text{and} \quad y_2 = x - 2$$

$$\therefore \quad y_1 - y_2 = -\frac{1}{x}.$$

$\therefore$ $y_1 > y_2$ for negative values of x

and $y_1 < y_2$ for positive values of x

$\therefore$ The curve lies above the asymptote when x is negative and below the asymptote when x is positive.

To investigate the position of the curve and the other asymptote $x = 0$, we suppose

$$x = \frac{a}{y} + \frac{b}{y^2} + \frac{c}{y^3} + \cdots\cdots$$

Putting this value of x in the equation to the curve

$$yx = x^2 - 2x - 1$$

we get

$$y\left(\frac{a}{y} + \frac{b}{y^2} + \frac{c}{y^3} + \cdots\right) = \left(\frac{a}{y} + \frac{b}{y^2} + \frac{c}{y^3} + \cdot\right)^2$$

$$-2\left(\frac{a}{y} + \frac{b}{y^2} + \frac{c}{y^3} + \cdots\right) - 1$$

or, $$a + \frac{b}{y} + \frac{c}{y^2} + \cdots = \frac{a^2}{y^2} + \cdots - 2\left(\frac{a}{y} + \frac{b}{y^2} + \frac{c}{y^3} + \cdots\right) - 1$$

Equating $a=-1$

$b=-2a$

$c=a^2-2b$

$\therefore \quad a=-1, b=2, c=1-4=-3$

$$\therefore \quad x=-\frac{1}{y}+\frac{2}{y^2}-\frac{3}{y^3}+\cdots$$

$$=-\frac{1}{y}, \text{ neglecting the smaller terms.}$$

$\therefore$ The difference between the abscissa of the curve and that of the asymptote $x=0$ for the same value of y is

$$x_1-x_2=-\frac{1}{y}$$

$\therefore \quad x_1>x_2$ for negative values of y

and $\quad x_1<x_2$ for positive values of y

$\therefore$ The curve lies on the right side of the asymptote $x=0$ when y is negative and on the left side of it when y is positive.

Theorem :

If $y=mx+c$ be the type of oblique asymptotes of the general rational algebraic equation.

$$u_n+u_{n-1}+u_{n-2}+\cdots+u_2+u_1+u_0$$

where $u_r=x^r\phi_r\left(\frac{y}{r}\right)$, $\phi_r\left(\frac{y}{r}\right)$ being a polynomial of degree r, at most, then

(*i*) Slope m of the asymptotes are given by the equation $\phi_n(m)=0$.

(*ii*) If m_1 be one of the root of $\phi_n'm)=0$, then c_1 corresponding to m_1 is obtained from

$$c_1\phi'_n(m_1)+\phi_{n-1}(m_1)=0 \quad \text{if } \phi'_n(m_1)\neq 0.$$

Proof : The given equation can be written as

$$x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right)+\cdots$$
$$\cdots+x\phi_1\left(\frac{y}{x}\right)+\ _0\left(\frac{y}{x}\right)=0 \qquad \cdots \quad (1)$$

Dividing by x^n

$$\phi_n\left(\frac{y}{x}\right)+\frac{1}{x}\phi_{n-1}\left(\frac{y}{x}\right)+\frac{1}{x^2}\phi_{n-2}\left(\frac{y}{x}\right)+\cdots$$
$$\cdot+\frac{1}{x^{n-1}}\phi_1\left(\frac{y}{x}\right)+\frac{1}{x^n}\phi_0\left(\frac{y}{x}\right)=0$$

Now proceeding to the limit as $x\to 0$ we get

$$\phi_n(m)=0 \qquad \cdots \qquad \cdots \ (2)$$

which gives the slopes of the asymptotes.

Solving (2) we shall get the different values of the slopes. If m_1 be one of its roots then the asymptote corresponding to m_1 is of the form

$$y=m_1x+p_1$$

i.e., $$\frac{y}{x}=m_1+\frac{p_1}{x}$$

Putting this in (1) we get

$$x^n\phi_n\left(m_1+\frac{p_1}{x}\right)+x^{n-1}\phi_{n-1}\left(m_1+\frac{p_1}{x}\right)+x^{n-2}\phi_{n-2}\left(m_1+\frac{p_1}{x}\right)+$$
$$\cdots+x\phi_1\left(m_1+\frac{p_1}{x}\right)+\phi_0\left(m_1+\frac{p_1}{x}\right)=0.$$

Applying Taylor's Theorem in each term

$$x^n\left\{\phi_n(m_1)+\frac{p_1}{x}\phi'_n(m_1)+\frac{p_1^2}{2x^2}\phi_n''(m_1)+\cdots\right\}$$
$$+x^{n-1}\left\{\phi_{n-1}(m_1)+\frac{p_1}{x}\phi'_{n-1}(m_1)+\frac{p_1^2}{2x^2}\phi''_{n-1}(m_1)+\cdots\right\}$$
$$+x^{n-2}\left\{\phi_{n-2}(m_1)+\frac{p_1}{x}\phi'_{n-2}(m_1)\right.$$
$$\left.+\frac{p_1^2}{2x^2}\phi''_{n-2}(m_1)+\cdots\right\}+\cdots=0$$

Arranging in the descending powers of x, we get

$$x^n\phi_n(m_1)+x^{n-1}\{p_1\phi'_n(m_1)+\phi_{n-1}(m_1)\}$$
$$+x^{n-2}\left\{\frac{p_1^2}{2}\phi''_n(m_1)+p_1\phi'_{n-1}(m_1)+\phi_{n-2}(m_1)\right\}+\cdots=0$$

As $\phi_n(m_1)=0$, this gives

$$x^{n-1}\{p_1\phi'_n(m_1)+\phi_{n-1}(m_1)\}$$
$$+x^{n-2}\left\{\frac{p_1^2}{2}\phi_n''(m_1)+p_1\phi'_{n-1}(m_1)+\phi_{n-2}(m_1)\right\}$$
$$+\cdots=0$$

or, $$p_1\phi'_n(m_1)+\phi_{n-1}(m_1)+\frac{1}{x}\left\{\frac{p_1^2}{2}\phi''_n(m_1)\right.$$
$$\left.+p_1\phi'_{n-1}(m_1)+\phi_{n-2}(m_1)\right\}+\cdots=0 \quad \cdots \quad (3)$$

Now proceeding to the limit as $x\to\infty$ and writing $Lt\ p_1=c_1$ we get

$$c_1\phi'_n(m_1)+\phi_{n-1}(m_1)=0$$
$$\Rightarrow \quad c_1=-\frac{\phi_{n-1}(m_1)}{\phi'_n(m_1)}, \text{ provided } \phi_n'(m_1)\neq 0$$

9.3. Working rule to find the asymptotes of an algebraic equation if $\phi'_n(m)\neq 0$.

From the given equation, find $\phi_n(m)=0$

Let m_1, m_2, $m_3\cdots$be its roots. Let $c_1, c_2, c_3\cdots$be the values of c corresponding to m_1, m_2, $m_3\cdots$

Then $$c_1=-\frac{\phi_{n-1}(m_1)}{\phi_n'(m_1)},$$
$$c_2=-\frac{\phi_{n-1}(m_2)}{\phi_n'(m_2)}$$
$$c_3=-\frac{\phi_{n-1}(m_3)}{\phi_n'(m_3)}$$

and so on.

Hence, the asymptotes are

$$y=m_1x-\frac{\phi_{n-1}(m_1)}{\phi'_n(m_1)}$$

$$y=m_2x-\frac{\phi_{n-1}(m_2)}{\phi'_n(m_2)}$$

$$y=m_3x-\frac{\phi_{n-1}(m_3)}{\phi'_n(m_3)} \text{ etc.}$$

Theorem :

If in an algebraic equation $\phi_n'(m_1)=0=\phi_{n-1}(m_1)$ then c_1 of asymptotes is obtained from the equation

$$\frac{c_1^2}{2}\phi_n''(m_1)+c_1\phi'_{n-1}(m_1)+\phi_{n-2}(m_1)=0.$$

If $\phi'_n(m_1)=0$, then

$$c_1=-\frac{\phi_{n-1}(m_1)}{\phi'_n(m_1)}$$

cannot give any value of c_1 corresponding to the slope m_1. So, in this case, there is no asymptote corresponding to the slope m_1.

If, however, both $\phi'_n(m_1)=0$, $\phi_{n-1}(m_1)=0$,

then equation (3) of Art 9·2 gives

$$\frac{1}{x}\left\{\frac{p_1^2}{2}\phi''_n(m_1)+p_1\phi'_{n-1}(m_1)+\phi_{n-2}(m_1)\right\}+\frac{1}{x^2}\{\cdots\}+\cdots=0$$

or, $$\frac{p_1^2}{2}\phi''_n(m_1)+p_1\phi'_{n-1}(m_1)+\phi_{n-2}(m_1)+\frac{1}{x}\{\cdots\}+\cdots=0.$$

So proceeding to the limit as $x\to\infty$ and writing $Lt\ p_1=c_1$, we get

$$\frac{c_1^2}{2}\phi''_n(m_1)+c_1\phi'_{n-1}(m_1)+\phi_{n-2}(m_1)=0.$$

This gives the two values c'_1 and c_1'' of c_1 corresponding to the slope m_1 provided $\phi''_n(m_1)\neq 0$.

Thus, the two asymptotes corresponding to the slope m_1 are

$$y=m_1x+c_1' \text{ and } y=m_1x+c_1''.$$

These are clearly parallel and such asymptotes are known as the case of parallel asymptotes.

Note. The equation $2x^3-x^2y+2xy^2+y^3-4x^2+8xy-4x+1=0$ can be put in to form

$$x^3\left\{2-\left(\frac{y}{x}\right)+2\left(\frac{y}{x}\right)^2+\left(\frac{y}{x}\right)^3\right\}+x^2\left\{-4+8\left(\frac{y}{x}\right)\right\}$$
$$+x\{-4\}+1=0.$$

So that

$$\phi_n(m)=2-m+2m^2+m^3$$
$$\phi_{n-1}(m)=-4+8m$$
$$\phi_{n-2}(m)=-4$$
$$\phi_{n-3}(m)=1$$

These show that the polynomial $\phi_n(m)$ might be written down at once by putting $x=1$ and $y=m$ in the highest degree terms of the original equation. $\phi_{n-1}(m)$, $\phi_{n-2}(m)$ and $\phi_{n-3}(m)$ might be obtained from the next successive lower degree terms in the similar way.

Ex. Obtain the asymptotes of the curve where equation is given by

$$\mathbf{x^3+2x^2y-4xy^2-8y^3-4x+8y-1=0} \qquad (C.\ H.\ 1965\ old)$$

Here $\phi_n(m)=1+2m-4m^2-8m^3$.

$$\phi_{n-1}(m)=0$$
$$\phi_{n-2}(m)=-4+8m$$

For oblique asymptotes $\phi_n(m)=0$

which gives $1+2m-4m^2-8m^3=0$

or, $(1+2m)-4m^2(1+2m)=0$

or, $(1+2m)^2(1-2m)=0$

$\Rightarrow$ $m=\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}$.

Again $c=-\dfrac{\phi_{n-1}(m)}{\phi'_n(m)}=\dfrac{0}{2-8m-24m^2}=0$ when $m=\dfrac{1}{2}$.

Also $c=\dfrac{0}{2+4-6}=$ undefined when $m=-\dfrac{1}{2}$.

So for $c=0$, one asymptote is $y=\dfrac{1}{2}x$

For other asymptotes c to be obtained from

$$\frac{c^2}{2}\phi''_n(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)=0.$$

which gives

$$\frac{c^2}{2}\{-8-48m\}+c\times0+(-4+8m)=0$$

or, $\dfrac{c^2}{2}\{-8+24\}-4-4=0$

or, $8c^2=8$ or, $c^2=1$

$\therefore$ $c=\pm1$.

So that the remaining two parallel asymptotes are

$y=-\frac{1}{2}x+1$ and $y=-\frac{1}{2}x-1$.

Hence, the three asymptotes are

$$x-2y=0$$
$$x+2y-2=0$$
$$x+2y+2=0.$$

9.4. Asymptotes parallel to the axes for an algebraic equation to a curve.

Arranging in descending powers of y, the equation takes the form

$$y^m\phi(x)+y^{m-1}\phi_1(x)+y^{m-2}\phi_2(x)+\cdots+\phi_m(x)=0 \quad \cdots \quad (1)$$

where $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)\cdots$are polynomials in x.

Dividing each term by y^m

$$\phi(x)+\frac{1}{y}\phi_1(x)+\frac{1}{y^2}\phi_2(x)+\cdots+\frac{1}{y^m}\phi_m(x)=0. \quad \cdots \quad (2)$$

Let $y \to 0$ and we write $\underset{x \to \infty}{Lt}\ x = k$

then from (2)

$$\phi(k) = 0$$

This shows that k is a root of $\phi(x) = 0$.

Thus if k_1, k_2, $k_3 \cdots$etc. are the real roots of $\phi(x) = 0$ then the real asymptotes parallel to the y axis will actually exist and the equation of the asymptotes will be $x = k_1$, $x = k_2$, $x = k_3 \cdots\cdots$all parallel to y axis.

Since $x = k_1$, k_2, $k_3 \cdots$etc. are the roots of $\phi(x) = 0$, we can write

$$\phi(x) = (x - k_1)(x - k_2)(x - k_3)\cdots\cdots$$

So from (1), we get

$$y^m\{(x - k_1)(x - k_2)\cdots\} + y^{m-1}\ \phi_1(x) + \cdots + \phi_m(x) = 0$$

This shows that the asymptotes parallel to y axis can be obtained by equating to zero the real linear factors in the coefficients of the highest power of y present in the given equation.

If the coefficient of the highest power of y is, however, a constant or is not resolvable into real linear factors, then the asymptotes parallel to y axis cannot exist.

Again arranging the given equation in descending powers of x, it can be shown as before that :

The asymptotes parallel to x axis can be obtained by equating to zero the real linear factor in the co-efficients of the highest power of x present in the given equation and no such asymptote can exist if the co-efficient of the highest power of x is a constant or is not resolvable into real linear factors.

Cases of failure of any parallel asymptotes.

In certain curves, it may happen that the above rules give no parallel or oblique asymptote. But little consideration will show that the said curve possesses no infinte branch. So in such a curve there cannot be any asymptote.

Ex. Examine the existence of asymptotes if any of the curve.

$$x^2y^2-2y+a^2=0. \qquad \text{(C. H. 1960, 68, 72)}$$

Here the coefficient of the highest power y^2 of y is x^2

So the asymptote parallel to y axis is $x=0$. (i.e., y axis)

But from the given equation

$$y^2=\frac{2y-a^2}{x^2}.$$

$$\therefore \quad y=\pm\frac{\sqrt{2y-a^2}}{x}.$$

$\Rightarrow$ y becomes imaginary when $x\to 0$

$\therefore$ There is no infinite branch of the curve along y axis and so y axis cannot be an asymptote.

Similary, it can be shown that there cannot be any infinte branch of the curve along x axis and so $y=0$ (i. e., x axis) cannot be the asymptote of the curve.

For oblique asymptote, we have $\phi_n(m)=m^2=0$ which shows that the curve cannot have also any oblique asymptote.

Hence, the given curve has neither parallel nor oblique asymptote.

Illustrative Examples :

Ex. 1. Find the asymptotes of the curve

$$x^2y^2-x^2y-xy^2+x+y+1=0 \qquad [\text{ C. H. 1948 }]$$

Since, degree of the equation is 4, there may be at most 4 asymptotes.

The co-efficient of the highest power of y is x^2-x

But $x^2-x=x(x-1)$

$\therefore$ $x=0$ and $x-1=0$ are the two asymptotes parallel to y axis.

Again, the co-efficient of the highest power of x is y^2-y

But $y^2-y=y(y-1)$

$\therefore$ $y=0$ and $y-1=0$ are the two asymptotes parallel to x axis.

Hence, the asymptotes are $x=0$, $y=0$, $x=1$, $y=1$.

Ex. 2. Find the asymptotes of the curve

$$\mathbf{y=\frac{x^2-2x-1}{x}} \qquad (\textit{C. H. 1967})$$

Equation to the curve is

$$x^2-xy-2x-1=0.$$

Since, the degree of the equation is 2, there may be at most 2 asymptotes.

The co-efficient of the highest power of x is 1 which is a constant. So there is no asymptote parallel to x axis.

The co-efficient of the highest power of y is $-x$ which is not a constant. So the asymptote parallel to y axis is $\quad -x=0$

i.e., $\quad x=0.$

Again, for the oblique asymptote, we write

$$\phi_n(m)=1-m, \qquad \phi_{n-1}(m)=-2.$$

If $y=mx+c$ be the asymptote, m is obtained from

$$\phi_n(m)=0 \quad \text{i e., from } 1-m=0 \quad \therefore \quad m=1.$$

c is obtained from $c=-\dfrac{\phi_{n-1}(m)}{\phi'_n(m)}=-\dfrac{-2}{-1}=-2$

$\therefore$ the oblique asymptote is $y=x-2$

Hence, the equation has two asymptotes

$$x=0 \text{ and } y=x-2.$$

9.5. An alternative method of finding the asymptotes of an algebraic curve.

The general algebraic curve of nth degree is of the form

$$(a_0y^n+a_1y^{n-1}x+a_2y^{n-2}x^2+\cdots\cdots+a_nx^n)$$
$$+(b_1y^{n-1}+b_2y^{n-2}x+\cdots+b_nx^{n-1})+\cdots+k_n=0. \qquad \cdots (1)$$

Let $y-m_1x$ be a non-repeated factor of the nth degree terms of the above equation. Then we can write

$$a_0y^n+a_1y^{n-1}x+\cdots+a_nx^n=(y-m_1x)F_{n-1}$$

where F_{n-1} contains terms only of degree $n-1$.

So equation (1) can be put in the form

$$(y-m_1x)F_{n-1}+P_{n-1}=0$$

where P_{n-1} contains terms of degree not greater than $n-1$.

$$\therefore \quad y-m_1x=-\frac{P_{n-1}}{F_{n-1}}. \quad \cdots \quad \cdots \quad \cdots \quad (2)$$

Since m_1 is a root of $\phi_n(m)=0$, the equation of the type $y=m_1x+c_1$ exists provided we can determine a finite value of c_1 from the formula,

$$c_1=\underset{x\to\infty}{Lt}(y-m_1x).$$

$\therefore$ from (2)

$$c_1=\underset{x\to\infty}{Lt}(y-m_1x)=\underset{x\to\infty}{Lt}-\frac{P_{n-1}}{F_{n-1}}.$$

Hence, the asymptote corresponding to the slope m_1 can be put in the form

$$y-m_1x \quad \underset{x\to\infty}{Lt}\left\{-\frac{P_{n-1}}{F_{n-1}}\right\}.$$

For any other non-repeated factor $y-m_2x$ of the nth degree terms the equation to the asymptote can be put in a similar way.

Illustrative Examples :

Ex. 1. Determine the asymptotes of the curve

$$\mathbf{x^4-x^2y^2+x^2+y^2-a^2=0.} \qquad (C.\ H.\ 1961)$$

Since, the equation is of degree 4 it has atmost 4 asymptotes.

The co-efficient of the highest power of y is $-x^2+1$ which is not constant.

So the equation has asymptotes $-x^2+1=0$

i.e., $x=\pm 1$ parallel to y axis.

For other two asymptotes, we write the equation as

$$x^2(x+y)(x-y)+x^2+y^2-a^2=0$$

$\therefore$ the asymptote parallel to $x-y=0$ is

$$x-y+\underset{\substack{x\to\infty \\ y=x}}{L}\ \frac{x^2+y^2-a^2}{x^3+x^2y}=0$$

or, $$x-y+Lt\ \frac{1+\left(\frac{y}{x}\right)^2-\frac{a^2}{x^2}}{x\left(1+\frac{y}{x}\right)}=0$$

or, $x-y=0$

The asymptote parallel to $x+y=0$ is

$$x+y+\underset{\substack{x\to\infty \\ y=-x}}{Lt}\ \frac{x^2+y^2-a^2}{x^3-x^2y}=0$$

or, $$x+y+\underset{\substack{x\to\infty \\ y=-x}}{Lt}\ \frac{1+\left(\frac{y}{x}\right)-\frac{a^2}{x^2}}{x\left(1-\frac{y}{x}\right)}=$$

or, $x+y=0.$

Hence, the four asymptotes are

$$x\pm 1=0,\ x\pm y=0$$

Ex. 2. Find the asymptotes, if any, of the curves

$(\mathbf{x}^2-\mathbf{y}^2)(\mathbf{x}+2\mathbf{y}-1)=3\mathbf{x}+4\mathbf{y}+5$ (*C. H.* 1966 **old**)

The asymptote parallel to $x-y=0$ is

$$x-y-\underset{\substack{x\to\infty \\ y=x}}{Lt}\ \frac{3x+4y+5}{(x+y)(x+2y-1)}=0$$

or, $x-y-\underset{x\to\infty}{Lt}\dfrac{7x+5}{2x(3x-1)}=0$

or, $x-y-\underset{x\to\infty}{Lt}\dfrac{7+\frac{5}{x}}{2x\left(3-\frac{1}{x}\right)}=0$

or, $x-y=0$

The asymptote parallel to $x+y=0$ is

$$x+y-\underset{\substack{x\to\infty\\ y=-x}}{Lt}\frac{3x+4y+5}{(x-y)(x+2y-1)}=0$$

or, $x+y-\underset{x\to\infty}{Lt}\dfrac{-x+5}{2x(-x-1)}=0$

or, $x+y=0$

The asymptote parallel to $x+2y-1=0$ is

$$x+2y-1-\underset{\substack{x\to\infty\\ y=-\frac{1}{2}x}}{Lt}\frac{3x+4y+5}{x^2-y^2}=0.$$

or, $x+2y-1-\underset{x\to\infty}{Lt}\dfrac{3x-2x+5}{x^2-\frac{1}{4}x^2}=0.$

or, $x+2y-1-\underset{x\to\infty}{Lt}\dfrac{1+\frac{5}{x}}{x\left(\frac{3}{4}\right)}=0$

or, $x+2y-1=0$

Hence, three asymptotes are

$$x\pm y=0,\quad x+2y-1=0.$$

9.6. Intersection of a curve and its asymptotes.

Theorem :

Any asymptote of a curve of the nth degree cuts the the curve in (n − 2) points.

Let the equation to the curve of nth degree be

$$x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right)+\cdots=0 \quad \cdots (1)$$

Let $y=mx+c$ be an oblique asymptote of (1),...(2)

Solving (1) and (2) we get

$$x^n\phi_n\left(m+\frac{c}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{c}{x}\right)+\cdots=0 \quad \cdots \quad (3)$$

So the co-ordinates of the point of intersection of (1) and (2) are given by (3).

Now expanding each term of (3) by Taylor's Theorem we get

$$x^n\left\{\phi_n(m)+\frac{c}{x}\phi'_n(m)+\frac{c^2}{x^2\lfloor 2}\phi''_n(m)+\cdots\right\}$$
$$+x^{n-1}\left\{\phi_{n-1}(m)+\frac{c}{x}\phi'_{n-1}(m)+\frac{c^2}{x^2\lfloor 2}\phi''_{n-1}(m)+\cdots\right\}+\cdots=0$$

$$\text{or,}\quad x^n\phi_n(m)+\{c\phi'_n(m)+\phi_{n-1}(m)\}x^{n-1}$$
$$+\{\tfrac{1}{2}c^2\phi''_n(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)\}x^{n-2}+\cdots=0 \cdots (4)$$

Now $y=mx+c$ is an asymptote

$\Rightarrow$The co-efficient of x^n and x^{n-1} are both zero.

So $y=mx+c$ is not an asymptote of the curve

$$\{\tfrac{1}{2}c^2\phi''_n(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)\}x^{n-2}$$
$$+\{\cdots\}x^{n-3}+\{\cdots\}x^{n-4}+\cdots=0 \quad \cdots \quad (5)$$

But (5) represents an equation of degree $(n-2)$ and which will give $(n-2)$ values of x.

Hence, any asymptote will cut the curve again in $(n-2)$ points.

Cor. 1. The n asymptotes of a curve of n degree will cut the curve in $n(n-2)$ points.

Cor. 2. Suppose an algebraic equation to a curve of the nth degree can be put in the form

$$F_n+F_{n-2}=0 \quad \cdots \quad (1)$$

where F consists of n non-repeated linear factors giving n asymptotes to the curve (1).

Hence, the $n(n-2)$ point of intersection of the asymptotes and the curve lie on the curve

$$F_{n-2}=0.$$

In particular, for a cubic curve $n=3$

$\therefore$ The 3 asymptotes will cut the curve in $3(3-2)$ i.e., 3 points which lie on the curve of degree $3-2$ i.e. 1 i.e., on a straight line.

For a quartic, $n=4$.

$\therefore$ The 4 asymptotes will cut the curve in $4\ (4-2)$ i.e., 8 points which lie on a curve of degree $4-2$ i.e. on a conic.

Illustrative Examples :

Ex. 1. Show that the point of intersection of the curve

$$x^4-5x^2y^2+4y^4+x^2-y^2+x+y+1=0$$

and its asymptotes lie on a conic. Identify this conic.

(C. H. 1966)

Let $y=mx+c$ be an asymptote.

From the given curve

$$\phi_n(m)=1-5m^2+4m^4$$

$$\phi_{n-1}(m)=0$$

$$\phi_{n-2}(m)=1-m^2.$$

For the asymptotes $\phi_n(m)=0$

$$\Rightarrow 1-5m^2+4m^4=0$$

$$\Rightarrow (1-m^2)(1-4m^2)=0$$

$$\therefore\quad m=1,\ -1,\ \tfrac{1}{2},\ -\tfrac{1}{2}$$

$$c=-\frac{\phi_{n-1}(m)}{\phi'_n(m)} \text{ for all } m$$

$$=\frac{0}{-10m+16m^3}$$

$$=0 \text{ for } m=1,\ -1,\ \tfrac{1}{2} \text{ and } -\tfrac{1}{2}$$

$\therefore$ the asymptotes are

$$y=x,\ y=-x,\ y=\tfrac{1}{2}x,\ y=-\tfrac{1}{2}x$$

i.e., $x-y=0$, $x+y=0$, $x+2y=0$, $x-2y=0$.

$\therefore$ Their joint equation is

$$(x-y)(x+y)(x+2y)(x-2y)=0$$

or, $x^4-5x^2y^2+4y^4=0$. $\cdots$ (1)

Now equation to the curve may be put in the form

$$(x^4-5x^2y^2+4y^4)+(x^2-y^2+x+y+1)=0.$$

i.e., $F_4+F_2=0$.

So the points of intersection of the curve and the asymptotes lie on the curve $F_2=0$

i.e., on $x^2-y^2+x+y+1=0$ which is a conic. $\cdots$ (2)

Equation (2) can be put in the form

$$\left(x+\frac{1}{2}\right)^2-\left(y-\frac{1}{2}\right)^2=-1 \text{ which is a hyperbola}$$

Hence, the points of intersection lie on the hyperbola

$$\left(x+\frac{1}{2}\right)^2-\left(y-\frac{1}{2}\right)^2=-1.$$

Ex. 2. Show that the intersection of the curve

$$\mathbf{2y^3-2x^2y-4xy^2+4x^3-14xy+6y^2+4x^2+6y+1=0}$$

and its asymptotes lie on the straight line $8x+2y+1=0$.

(C. H. 1964, 73)

Let $y=mx+c$ be an asymptote.

From the given curve

$$\phi_n(m)=2m^3-2m-4m^2+4$$

$$\phi_{n-1}(m)=-14m+6m^2+4$$

Now $\phi_n(m)=0$ gives $m^3-2m^2-m+2=0$

or, $m^2(m-2)-(m-2)=0$

$\therefore$ $m=1, -1, 2.$

Also $c=-\dfrac{\phi_{n-1}(m)}{\phi'_n(m)}$ for all m

$$=-\frac{-14m+6m^2+4}{6m^2-2-8m}$$

$$=\frac{7m-3m^2-2}{3m^2-4m-1}$$

$= -1,\ -2$ or 0 according as $m=1,\ -1$ or 2

$\therefore$ the asymptotes are

$$x-y-1=0,\ x+y+2=0 \text{ and } 2x-y=0.$$

So, their joint equation is

$$(x-y-1)(x+y+2)(2x-y)=0.$$

or, $y^3 - x^2y-2xy^2+2x^3-7xy+3y^2+2x^2+2y-4x=0$

multiplying this equation by 2 and subtracting from the given equation we, get $8x+2y+1=0$.

Hence, the asymptotes lie on the straight line

$$8x+2y+1=0.$$

Ex. 3. Determine the asymptotes of the curve

$$4(x^4+y^4)-17x^2y^2-4x(4y^2-x^2)+2(x^2-2)=0$$

and show that they pass through the points of intersection of the curve and the ellipse $x^2+4y^2=4$ (C. H. 1962)

Let $y=mx+c$ be an asymptote.

$\therefore\quad \phi_n(m)=4m^4-17m^2+4$

$\phi_{n-1}(m)=-16m^2+4$

Now $\phi_n(m)=0$ gives $4m^4-17m^2+4=0$

$4m^2-16m^2-m^2+4=0$

$(4m^2-1)(m^2-4)=0$

$\therefore\quad m^2=4, \frac{1}{4}$

$\therefore\quad m=\pm 2,\ \pm\frac{1}{2}$

Also $\quad c=-\dfrac{\phi_{n-1}(m)}{\phi'_n(m)}=\dfrac{16m^2-4}{16m^3-34m}$

$=1,\ -1,\ 0,\ 0$ for $m=2,\ -2,\ \frac{1}{2},-\frac{1}{2}$

$\therefore$ The asymptotes are

$y=2x+1,\ y=-2x-1,\ 2y=x,\ 2y=-x$

So, their joint equation is

$$(2y+x)(2y-x)(y+2x+1)(y-2x-1)=0.$$

$$\text{or,}\quad 4(x^4+y^4)-17x^2y^2-4x(4y^2-x^2)+x^2-4y^2=0.$$

Subtracting this equation from the given equation, we get $x^2+4y^2=4$.

Hence, the point of intersection lie on the ellipse

$$x^2+4y^2=4.$$

9.7. Asymptote for Polar equation.

Theorem :

In polar equation $r=f(\theta)$, if $\theta \to \alpha$ as $r \to \infty$ and if $f(\theta)\sin(\theta-\alpha) \to p$, then the line $r\sin(\theta-\alpha)=p$ is an asymptote of the curve.

Let $u=\frac{1}{r}=\frac{1}{f(\theta)}=F(\theta)$ (say).

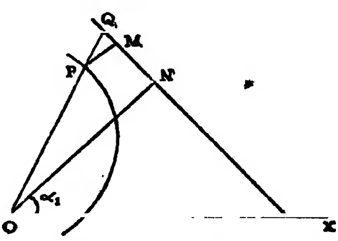
Let $P(r, \theta)$ be any point on the curve. If $P \to \infty$ as $r \to \infty$, then $F(\theta) \to 0$.

Let the solutions of $F(\theta)=0$ be

$$\theta=\alpha_1, \alpha_2, \alpha_3 \quad \cdots$$

then as the radius vector make angles $\alpha_1, \alpha_2, \alpha_3 \cdots$ the branches of the curve tends to infinity.

Let us consider the infinite branch when $\theta=\alpha_1$ and let $r\cos(\theta-\alpha_1)=p \cdots (1)$ be the asymptote of this branch.



Let QN be a line so that the perpendicular ON from O on the line make an angle $NOX=\alpha_1$. Let OP produced to meet this line at Q. PM is drawn perpendicular to QN.

Then $\quad PM = PQ \cos QPM$

$$= (OQ - OP) \cos QON$$

$$= (OQ - OP) \cos (\theta - \alpha_1)$$

$$= \{p \sec (\theta - \alpha_1) - f(\theta)\} \cos (\theta - \alpha_1)$$

$$= p - f(\theta) \cos (\theta - \alpha_1) \quad \cdots \quad \cdots \quad (2)$$

Equation (1) is an asymptote

$$\Rightarrow p \to \infty,\ \theta \to \alpha_1 \text{ and } PM \to 0$$

$\therefore$ From (2) $\quad 0 = p - \underset{\theta \to \alpha_1}{Lt} \left\{ f(\theta) \cos (\theta - \alpha_1) \right\}$

or, $\quad \underset{\theta \to \alpha_1}{Lt} f(\theta) \cos (\theta - \alpha_1) = p$

But when $f(\theta) \to \propto$ as $\theta \to \alpha_1$, p remains finite, we must have

$$\underset{\theta \to \propto}{Lt} \cos (\theta - \alpha_1) = 0$$

$$\therefore \quad \theta - \alpha_1 = \frac{\pi}{2}.$$

Now $\quad p = \underset{\theta \to \propto}{Lt} f(\theta) \cos (\theta - \alpha_1)$

$$= \underset{\theta \to \propto}{Lt} \frac{\cos (\theta - \alpha_1)}{F(\theta)} \text{ form } \frac{0}{0}$$

$$= \underset{\theta \to \propto}{Lt} \frac{-\sin (\theta - \alpha_1)}{F'(\theta)}$$

$$= \frac{1}{F'(\alpha_1)}$$

So the asymptotes to the curve becomes

$$r \cos \left(\theta - \alpha_1 + \frac{\pi}{2} \right) = -\frac{1}{F'(\alpha_1)}$$

or, $\quad r \sin (\theta - \alpha_1) = \dfrac{1}{F'(\alpha_1)}$

Similarly, asymptotes corresponding to $\theta = \alpha_2, \alpha_3$ etc. can be obtained.

Ex. Find the asymptote of

$$r\theta = a \qquad \text{(C. P. 1937)}$$

$$\because \quad \frac{1}{r} = \frac{\theta}{a}, \quad F(\theta) = \frac{\theta}{a}$$

$$\therefore \quad F'(\theta) = \frac{1}{a}$$

When $r \to \infty$, $\theta \to 0$

$\therefore$ Equation to the asymptote is

$$r \sin \theta = a$$

Exercise 9

1. Show that the asymptotes of the curve $xy - x - y = 0$ are $x - 1 = 0$ and $y - 1 = 0$.

2. Show that $y = \pm \frac{b}{a} x$ are the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

3. Find the asymptotes of

$$y^3 + x^2 y + 2xy^2 - y + 1 = 0. \qquad (C. P, 1941, 44)$$

(Ans. $x + y = \pm 1$, $y = 0$)

4. Find the asymptotes of

$$x^2(x-y)^2 - a^2(x^2 + y^2) = 0. \qquad (C. P. 1945)$$

(Ans. $x = \pm a$, $x - y = \pm a\sqrt{2}$)

5. Find the asymptotes of the cubic

$$x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0 \qquad (C. P. 1949)$$

(Ans. $x - y = 0$, $x + y + 1 = 0$, $x + 2y - 1 = 0$)

6. Find the asymptotes of

$$y^3 - xy^2 - x^2 y + x^3 + x^2 - y^2 - 1 = 0 \qquad (C. P. 1949)$$

(Ans. $y = \pm x$, $y = x + 1$)

7. Show that the curve

$$x^2(x^2 + y^2) - a^2(x^2 - y^2) = 0$$

has no asymptote.

8. Show that the curve

$$x^2(x - y) + ay^2 = 0$$

has only one asymptote $y = x + a$.

9. A straight line parallel to the asymptote of $x^2(x-y)+ay^2=0$ cut the curve in P and Q. Show that the mid point of PQ lies on the hyperbola $x(x-y)+ay=0$.

10. Show that the asymptotes of the curve

$$x^2y^2-a^2(x^2+y^2)-a^3(x+y)+a^4=0$$

form a square two of whose angular points lie on the curve.

(*C. P.* 1947)

11. Find the asymptotes of

$$y=\frac{x^3-3x^2+1}{x^2}$$

and investigate the position of the curve and the asymptotes.

(Ans. $x=0$, $y=x-3$.

12. Show that the curve $x^3+y^3=3axy$ lies above its asymptote $x+y-a=0$ both for positive as well as negative values of x.

13. Show that the curve $y^2=x^2(y-x)$ lies above or below its asymptote $y=x+1$ according as x is positive or negative.

14. Find the asymptotes of the curve

$$x^2y+xy^2-xy-2y^2-y+x=0$$

and show that they cut the curve again in three points which lie on the line $x+y=0$.

15. Show that the asymptotes of

$$(x^2-4y^2)(x^2-9y^2)+5x^2y-5xy^2-30y^3+xy+7y^2-1=0$$

cut the curve in eight points which lie on the circle $x^2+y^2=1$.

16. Find the equation of the cubic which has the same asymptote as the curve

$x^3-2x^2y-xy^2+2y^3+x+y+1=0$ and which passes through (0, 0), (1, 0) and (0, 1).

(Ans. $x^3-2x^2y-xy^2+2y^3-x-2y=0$).

17. Show that the curve $x^3+y^3=3ax^2$ lies above or below its asymptote $x+y-a=0$ according as x is positive or negative.

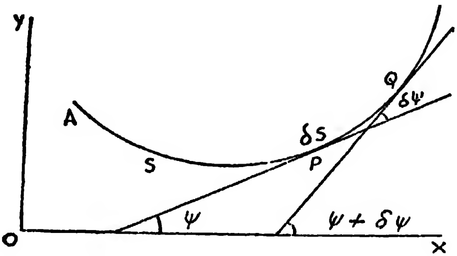
Also show that the equation of the tangent to the curve parallel to its asymptote is $x+y=4a$.

CHAPTER X

CURVATURE

10.1. Definition :

The curvature at a point of a curve is the rate at which the curve bends at that point.



Let P be a point on the curve $y=f(x)$ at an arc distance s from some fixed point A on the curve. Let Q be a point on the curve very near to P at an arc distance $s+\partial s$ from A so that arc $PQ=\partial s$. Let the tangent at P and Q make angles ψ and $\psi+\partial\psi$ with the x axis. Then $\partial\psi$ is the angle between the two tangent at P and Q and this measures the change of the direction of the tangents as the curve bends from P to Q.

So the average curvature of the arc PQ is $\dfrac{\partial\psi}{\partial s}$ and curvature at $P=\underset{\partial s\to 0}{Lt}\dfrac{\delta\psi}{\delta s}=\dfrac{d\psi}{ds}$.

Thus if k denotes the curvature at P, then

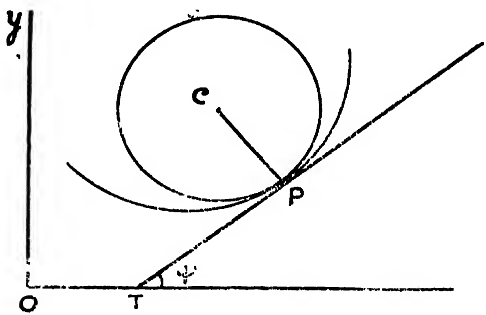
$$k=\frac{d\psi}{ds}.$$

The reciprocal of k is called the radius of curvature at P and is denoted by ρ, so that

$$\rho=\frac{ds}{d\psi}.$$

10.2. Centre of Curvature

Let P be any point on the curve $y=f(x)$ at which the radius of curvature is ρ. Measure a length $PC=\rho$ along the positive direction of the normal at P of the curve. Then the point C is called the centre of curvature and the circle drawn with C as centre and radius equal to $PC\ (=\rho)$ is called the circle of curvature at P.



Since PC is the normal to both the curve and the circle at P, the curvature of both the curve and the circle at P is the same. Hence, the radius of curvature at any point P of a curve is equal to the radius of the circle drawn through P whose curvature at P is the same as that of the curve at P.

Note. The formula $\rho=\dfrac{ds}{d\psi}$ is useful if the intrinsic equation of the curve be given. Thus if

$s=c \log \tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)$

Then $\dfrac{ds}{d\psi}=c \cdot \dfrac{1}{\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)} \cdot \sec^2\left(\frac{\pi}{4}+\frac{\psi}{2}\right) \cdot \frac{1}{2}$

$$=\frac{c}{2} \cdot \frac{1}{\sin\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\psi}{2}\right)}=\frac{c}{\sin\left(\frac{\pi}{2}+\psi\right)}=\frac{c}{\cos\psi}$$

$$=c\sec\psi.$$

Hence $\rho=\dfrac{ds}{d\psi}=c\sec\psi.$

10.3. Cartesian formula for radius of curvature.

Let $P(x, y)$ be a point on the curve $y=f(x)$ and let the tangent at P make an angle ψ with the x-axis so that

$$\frac{dy}{dx}=\tan\psi.$$

Differentiating w. r. to x

$$\frac{d^2y}{dx^2}=\sec^2\psi\frac{d\psi}{dx}$$

$$=\sec^2\psi\frac{d\psi}{ds}\cdot\frac{ds}{dx}$$

$$=\sec^3\psi\frac{d\psi}{ds} \qquad \left[\because \frac{dx}{ds}=\cos\psi\right]$$

$$\therefore \quad \rho=\frac{ds}{d\psi}$$

$$=\frac{\sec^3\psi}{\frac{d^2y}{dx^2}}=\frac{\{\sec^2\psi\}^{3/2}}{\frac{d^2y}{dx^2}}=\frac{\{1+\tan^2\psi\}^{3/2}}{\frac{d^2y}{dx^2}}$$

$$=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}.$$

$$=\frac{(1+y_1^2)^{3/2}}{y_2} \text{ provided } y_2\neq 0.$$

This formula fails at a point if $\frac{dy}{dx}$ at the point is infinite i.e., if $\psi=90^\circ$ i.e. if the tangent at the point is parallel to y-axis. In such a case, we write

$$\frac{dx}{dy}=\cot\psi$$

Differentiating w. r. to y

$$\frac{d^2x}{dy^2}=-\operatorname{cosec}^2\psi\frac{d\psi}{dy}$$

$$=-\operatorname{cosec}^2\frac{d\psi}{ds}\cdot\frac{ds}{dy}$$

$$=-\operatorname{cosec}^3\psi\frac{d\psi}{ds} \qquad \left[\because \frac{dy}{as}=\sin\psi\right]$$

$$\therefore \quad \rho=\frac{ds}{d\psi}$$

$$=\frac{\operatorname{cosec}^3\psi}{\frac{d^2x}{dy^2}}$$ [Considering the magnitude of ρ only so neglecting negative sign.]

$$=\frac{\{1+\cot^2\psi\}^{3/2}}{\frac{d^2x}{dy^2}}$$

$$=\frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}}$$

$$=\frac{(1+x_1{}^2)^{3/2}}{x_2}, \text{ provided } x_2\neq 0.$$

Ex. Find the radius of curvature of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at an extremity of the major axis.

Differentiating the given equation w. r. to x

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \quad \text{or.} \quad \frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

$\because$ At extremity of the major axis $y=0$.

$\dfrac{dy}{dx}$ at that point is infinite and so formula

$$\rho=\frac{(1+y_1{}^2)^{3/2}}{y_2} \text{ fails here.}$$

Again, differentiating the given equation w. r. to y

$$\frac{2x}{a^2}\frac{dx}{dy}+\frac{2y}{b^2}=0$$

$$\therefore \quad \frac{dx}{dy}=-\frac{a^2y}{b^2x}=0 \text{ at } (a, 0)$$

$$\frac{d^2x}{dy^2}=-\frac{a^2}{b^2}\frac{x-\dfrac{dx}{dy}\cdot y}{x^2}$$

$$=-\frac{a^2}{b^2}\cdot\frac{a}{a^2}=-\frac{a}{b^2}$$

$$\therefore \quad \rho=\frac{(1+x_1{}^2)^{3/2}}{x_2}=\frac{1}{a/b^2}=\frac{b^2}{a}.$$

10.4. Formula for radius of curvature from the parametric equation $x=f(t)$, $y=\phi(t)$. *(C. H. 1960 old)*

$$\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{\phi'(t)}{f'(t)}.$$

$$\frac{d^2y}{dx^2}=\frac{\phi''(t)f'(t)-f''(t)\phi'(t)}{\{f'(t)\}^2}\cdot\frac{dt}{dx}$$

$$=\frac{\phi''(t)f'(t)-f''(t)\phi'(t)}{\{f'(t)\}^3}$$

$$\therefore \quad \rho=\frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}{\dfrac{d^2y}{dx^2}}$$

$$=\frac{[\{f'(t)\}^2+\{\phi'(t)\}^2]^{3/2}}{\phi''(t)f'(t)-f''(t)\phi'(t)}.$$

$$=\frac{(x'^2+y'^2)^{3/2}}{y''x'-x''y'} \text{ provided } y''x'-x''y'\neq 0.$$

Illustrative Examples :

Ex. 1. If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2=4ax$ then

$$\rho_1^{-2/3}+\rho_2^{-2/3}=(2a)^{-2/3}.$$

Let PSp be any focal chord of the parabola. Let $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ be the co-ordinates of P and p so that

$$t_1t_2=-1.$$

Let $\quad x=at_1^2, \quad y=2at_1$

then $\quad x'=2at_1 \quad y'=2a$

$\therefore \quad x''=2a, \quad y''=0$

$$\therefore \quad \rho_1=\frac{(x'^2+y'^2)^{3/2}}{y''x'-x''y'}=\frac{(4a^2t_1^2+4a^2)^{3/2}}{-2a \,.\, 2a}$$

$$=(4a^2)^{\frac{1}{2}}(1+t_1^2)^{3/2} \quad \text{(neglecting negative sign)}$$

$$=2a(1+t_1^2)^{3/2}$$

Similarly, $\rho_2=2a(1+t_2^2)^{3/2}$

$$\therefore \quad \rho_1^{-2/3}+\rho_2^{-2/3}=\frac{(2a)^{-2/3}}{1+t_1^2}+\frac{(2a)^{-2/3}}{1+t_2^2}$$

$$=(2a)^{-2/3}\left\{\frac{1}{1+t_1^2}+\frac{1}{1+t_2^2}\right\}$$

$$=(2a)^{-2/3}\left\{\frac{1}{1+t_1^2}+\frac{t_1^2}{1+t_1^2}\right\} \quad \left[\because \quad t_2=-\frac{1}{t_1}.\right]$$

$$=(2a)^{-2/3}$$

Ex. 2. Find the radius of curvature of the cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ at any point on the curve.

(C. H. 1960 old)

Here, $\quad x'=a(1+\cos\theta), \; y'=a\sin\theta$

$\quad x''=-a\sin\theta, \quad y''=a\cos\theta.$

$$\therefore \quad \rho=\frac{(x'^2+y'^2)^{3/2}}{y''x'-x''y'}=\frac{\{a^2(1+2\cos\theta+\cos^2\theta)+a^2\sin^2\theta\}^{3/2}}{a\cos\theta.a(1+\cos\theta)+a\sin\theta \;\; a\sin\theta}$$

$$=a\frac{\{2(1+\cos\theta)\}^{3/2}}{\cos\theta+\cos^2\theta+\sin^2\theta}=\frac{a.2^{3/2}(1+\cos\theta)^{3/2}}{1+\cos\theta}$$

$$=2^{3/2}a(1+\cos\theta)^{\frac{1}{2}}$$

✓ **Ex. 3. Find the radius of curvature of**

$$x^{2/3}+y^{2/3}=a^{2/3} \text{ at } (x, y) \qquad \text{(C. H. 1960)}$$

Let $x=a\cos^3\theta, \qquad y=a\sin^3\theta.$

Then $x'=3a\cos^2\theta.(-\sin\theta)=-3a\cos^2\theta\sin\theta.$

$y'=3a\sin^2\theta\cos\theta.$

$\therefore\ x''=-3a(-2\cos\theta\sin^2\theta+\cos^3\theta)$

$y''=3a(2\sin\theta\cos^2\theta-\sin^3\theta)$

$$\therefore\ \rho=\frac{(x'^2+y'^2)^{3/2}}{y''x'-x''y'}$$

$$=\frac{(9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta)^{3/2}}{3a(2\sin\theta\cos^2\theta-\sin^3\theta)(-3a\cos^2\theta\sin\theta)+3a(-2\cos\theta\sin^2\theta+\cos^3\theta)\,3a\sin^2\theta\cos\theta}$$

$$=\frac{\{9a^2\sin^2\theta\cos^2\theta\}^{3/2}}{9a^2(-2\sin^2\theta\cos^4\theta+\cos^2\theta\sin^4\theta)+9a^2(-2\cos^2\theta\sin^4\theta+\sin^2\theta\cos^4\theta)}$$

$$=\frac{(3a\sin\theta\cos\theta)^3}{a^2\{-18\sin^2\theta\cos^4\theta+9\cos^2\theta\sin^4\theta-18\cos^2\theta\sin^4\theta+9\sin^2\theta\cos^4\theta\}}$$

$$=\frac{(3a\sin\theta\cos\theta)^3}{a^2\{-18\sin^2\theta\cos^2\theta+9\sin^2\theta\cos^2\theta\}}$$

$$=\frac{(3a\sin\theta\cos\theta)^3}{9a^2\sin^2\theta\cos^2\theta} \qquad [\because\ xy=a^2\sin^3\theta\cos^3\theta$$

$$=3a\sin\theta\cos\theta \qquad axy=a^3\sin^3\theta\cos^3\theta$$

$$=3(axy)^{1/3}. \qquad \therefore\ a\sin\theta\cos\theta=(axy)^{\frac{1}{3}}\ ]$$

Ex. 4. If ρ_1 and ρ_2 be the radii of curvature at the ends P and D of conjugate diameter of the ellipse $x^2/a^2+y^2/b^2=1$ then

$$(ab)^{2/3}(\rho_1^{2/3}+\rho_2^{2/3})=a^2+b^2. \qquad (C. H. 1966)$$

Let ϕ be the eccentric angle of P. Then the co-ordinates of P and D are respectively

$(a\cos\phi, b\sin\phi)$ and $(-a\sin\phi, b\cos\phi)$.

Now considering ρ_1 at $P(a\cos\phi, b\sin\phi)$

$x=a\cos\phi, \qquad y=b\sin\phi$

Then $x' = -a \sin \phi, \quad y' = b \cos \phi$

$x'' = -a \cos \phi, \quad y'' = -b \sin \phi$

$$\therefore \quad \rho_1 = \frac{(x'^2 + y'^2)^{3/2}}{y''x' - x''y'} = \frac{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{3/2}}{ab}$$

$$= \frac{(CD^2)^{3/2}}{ab} = \frac{CD^3}{ab}.$$

Similarly, ρ_2 at $D(-a \sin \phi, b \cos \phi)$

$$= \frac{CP^3}{ab}.$$

$$\therefore \quad \rho_1^{2/3} + \rho_2^{2/3} = \frac{CD^2 + CP^2}{(ab)^{2/3}} = \frac{a^2 + b^2}{(ab)^{2/3}}.$$

or, $(ab)^{2/3}(\rho_1^{2/3} + \rho_2^{2/3}) = a^2 + b^2.$

10.5. Radius of curvature for Implicit function f(x, y)=0.

$$\frac{dy}{dx} = -\frac{f_x}{f_y}, \; f_y \neq 0. \qquad \textbf{(C. H. 1969, B. H. 1968)}$$

$$\Rightarrow f_x + f_y \frac{dy}{dx} = 0$$

Differentiating w. r. to x

$$f_{xx} + f_{xy} \frac{dy}{dx} + \left(f_{yx} + f_{yy} \frac{dy}{dx}\right)\frac{dy}{dx} + f_y \frac{d^2y}{dx^2} = 0.$$

$\because \quad f_{xy} = f_{yx}$, this gives

$$f_{xx} + 2f_{xy} \frac{dy}{dx} + f_{yy}\left(\frac{dy}{dx}\right)^2 + f_y \frac{d^2y}{dx^2} = 0.$$

$$\text{or,} \quad f_y \frac{d^2y}{dx^2} = -\left\{f_{xx} + 2f_{xy}\left(-\frac{f_x}{f_y}\right) + f_{yy} \frac{f_x^2}{f_y^2}\right\}$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{f_y^3}$$

$$\text{Now} \quad \rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}, \quad \frac{d^2y}{dx^2} \neq 0$$

$$= \frac{\{f_x^2 + f_y^2\}^{3/2}}{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}$$

when $f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2 \neq 0.$

Ex. Find the radius of curvature of the curve

$$\mathbf{x^2(x^2+y^2)+4ax^2y-2a^2x^2+3a^2y^2-4a^3y+a^4=0}$$

at the point (0, a). **(C. H. 1957)**

Let $f(x, y)=x^2(x^2+y^2)+4ax^2y-2a^2x^2+3a^2y^2-4a^3y+a^4=0.$

$$f_x=4x^3+2xy^2+8axy-4a^2x$$

$$\therefore \quad (f_x)_{0,a}=0.$$

$$f_y=2x^2y+4ax^2+6a^2y-4a^3$$

$$\therefore \quad (f_y)_{0,a}=6a^3-4a^3=2a^3.$$

$$f_{xx}=12x^2+2y^2+8ay-4a^2$$

$$\therefore \quad (f_{xx})_{0,a}=2a^2+8a^2-4a^2=6a^2$$

$$f_{xy}=4xy+8ax$$

$$\therefore \quad (f_{xy})_{0,a}=0$$

$$f_{yy}=2x^2+6a^2$$

$$\therefore \quad (f_{yy})_{0,a}=6a^2$$

Now $$\rho=\frac{\{f_x^2+f_y^2\}^{3/2}}{f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2}$$

$$=\frac{\{(2a^3)^2\}^{3/2}}{6a^2(2a^3)^2}=\frac{8a^9}{24a^8}=\frac{1}{3}a.$$

10.6. Newtonian Method.

If a curve passes through the origin and the axis of x is the tangent to the curve at the origin, then the radius of curvature at the origin is

$$\rho=\underset{x\to 0}{Lt}\ \frac{x^2}{2y}.$$

At the origin $x=0$, $y=0$.

$\therefore$ By Maclaurin's Theorem at (0, 0)

$$f(x)=f(0)+xf'(0)+\frac{x^2}{\underline{|2}}f''(0)+ \quad \cdots \qquad \cdots \qquad (1)$$

Also at the origin $f(0)=0$ and since tangent is the x axis, its slope at the origin $=0$

$\therefore \quad f'(0)=0$

$\therefore$ from (1), we get

$$y=\frac{x^2}{2}f''(0)+\frac{x^3}{6}f'''(0)+\cdot$$

or, $\frac{2y}{x^2}=f''(0)+\frac{x}{3}f'''(0)+$higher powers of x

$$\therefore \quad \underset{x\to 0}{Lt}\ \frac{2y}{x^2}=f''(0).$$

Now $\rho=\frac{[1+\{f'(0)\}^2]^{3/2}}{f''(0)}.$

$$=\frac{1}{f''(0)}=\underset{x\to 0}{Lt}\ \frac{x^2}{2y}.$$

Similarly, it can be shown that at the origin where y axis is the tangent

$$\rho=\underset{y\to 0}{Lt}\ \frac{y^2}{2x}.$$

10.7. Curvature at the origin of the curve

$$a_1x+a_2y+b_1x^2+b_2xy+b_3y^2+c_1x^3+\cdots=0$$

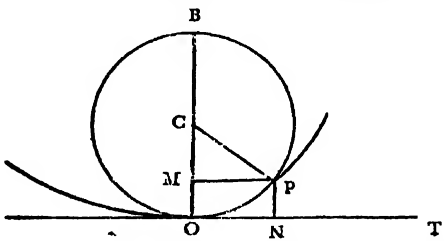
(C. H. 1964 old)

The tangent at the origin is

$$a_1x+a_2y=0, \quad (a_1\neq 0,\ a_2\neq 0)$$

Consider a circle touching the given curve along the curve OP where O is the origin and P is a neighbouring point.

PN is drawn perpendicular to the tangent OT at O.



Then PN = length of the perpendicular from $P(x, y)$ on $a_1x+a_2y=0$

$$=\frac{a_1x+a_2y}{\sqrt{a_1^2+a_2^2}}$$

Now from plane geometry, we have

$$OM\,.\,MB=PM^2$$

or, $OM(OB-OM)=PM^2$

or, $OM\;OB=PM^2+OM^2=OP^2$

$$\therefore\quad OB=\frac{OP^2}{OM}=\frac{OP^2}{PN}$$

$$=\frac{x^2+y^2}{(a_1x+a_2y)/\sqrt{a_1{}^2+a_2{}^2}}$$

But $OB\to 2\rho$ when $x\to 0$

$$\therefore\quad 2\rho=\underset{x\to 0}{Lt}\ \sqrt{a_1{}^2+a_2{}^2}\ \frac{x^2+y^2}{a_1x+a_2y}$$

$$\therefore\quad \rho=\tfrac{1}{2}\sqrt{a_1{}^2+a_2{}^2}\ \underset{x\to 0}{Lt}\left(\frac{x^2+y^2}{a_1x+a_2y}\right)$$

The curvature is the inverse of this.

Illustrative Examples :

Ex. 1. Find the radii of curvature of the curve $y^2-3xy-4x^2+x^3+x^4y+y^5=0$ at the origin. (C. H. 1958)

The tangents at the origin are

$$y^2-3xy-4x^2=0$$

or, $(y-4x)(y+x)=0$

i.e., $y-4x=0$ and $y+x=0$.

Now considering the tangent $y-4x=0$

$$\rho=\tfrac{1}{2}\sqrt{a^2+b^2}\ \underset{x\to 0}{Lt}\ \frac{x^2+y^2}{ax+by}$$

$$=\tfrac{1}{2}\sqrt{1^2+4^2}\ \underset{x\to 0}{Lt}\ \frac{x^2+y^2}{y-4x}$$

$$=\tfrac{1}{2}\sqrt{17}\ \underset{x\to 0}{Lt}\ \frac{(x^2+y^2)(y+x)}{-x^3-x^4y-y^5}\text{, from the given equation}$$

$$=\tfrac{1}{2}\sqrt{17}\ \underset{x\to 0}{Lt}\ \frac{\left\{1+\left(\frac{y}{x}\right)^2\right\}.\left(1+\frac{y}{x}\right)}{-1-xy-\frac{y^5}{x^3}}$$

$$=\tfrac{1}{2}\sqrt{17}\ \underset{x\to 0}{Lt}\ \frac{\left\{1+\left(\frac{y}{x}\right)^2\right\}\left(1+\frac{y}{x}\right)}{-1-x^2\left(\frac{y}{x}\right)-x^2\left(\frac{y}{x}\right)^t}$$

$$=\tfrac{1}{2}\sqrt{17}\ \underset{x\to 0}{Lt}\ \frac{(1+16)(1+4)}{-1-4x^2-4^5x^2}\qquad \left[\because\ y=4x\quad \therefore\ \frac{y}{x}=4\right]$$

$=\tfrac{1}{2}\sqrt{17\cdot 85}$, neglecting negative sign

$$=\frac{85\sqrt{17}}{2}.$$

Considering the tangent $y+x=0$

$$\rho=\tfrac{1}{2}\sqrt{1^2+1^2}\ \underset{x\to 0}{Lt}\ \frac{x^2+y^2}{y+x}$$

$$=\frac{\sqrt{2}}{2}\ \underset{x\to 0}{Lt}\ \frac{\left\{1+\left(\frac{y}{x}\right)^2\right\}\left(\frac{y}{x}-4\right)}{-1+x^2\left(\frac{y}{x}\right)-x^2\left(\frac{y}{x}\right)^t}$$

$$=\frac{\sqrt{2}}{2}\,\frac{(1+1)(-1-4)}{-1}=5\sqrt{2}.\qquad [\because\ y/x=-1]$$

Ex. 2. Find the curvature at the origin of each of the two branches of the curve

$$y(ax+by)=cx^3+ex^2y+fxy^2+gy^3\qquad \textbf{(C. H. 1961)}$$

Tangents at the origin of the two branches of the curve are

$$y=0 \text{ and } ax+by=0.$$

Considering the tangent $ax+by=0$

$$\rho=\tfrac{1}{2}\sqrt{a^2+b^2}\ \underset{x\to 0}{Lt}\ \frac{x^2+y^2}{ax+by}$$

$$=\tfrac{1}{2}\sqrt{a^2+b^2}\ \underset{x\to 0}{Lt}\ \frac{(x^2+y^2)y}{cx^3+ex^2y+fxy^2+gy^3}$$

$$=\tfrac{1}{2}\sqrt{a^2+b^2}\ \underset{x\to 0}{Lt}\ \frac{\left\{1+\left(\frac{y}{x}\right)^2\right\}\frac{y}{x}}{c+e\left(\frac{y}{x}\right)+f\left(\frac{y}{x}\right)^2+g\left(\frac{y}{x}\right)^3}$$

$$=\tfrac{1}{2}\sqrt{a^2+b^2}\;\frac{\left(1+\frac{a^2}{b^2}\right)\left(-\frac{a}{b}\right)}{c+e\left(-\frac{a}{b}\right)+f\left(-\frac{a}{b}\right)^2+g\left(-\frac{a}{b}\right)^3}$$

$$=-\tfrac{1}{2}\frac{(a^2+b^2)^{\frac{3}{2}}.a}{cb^3-eab^2+fa^2b-ga^3}$$

$$=\frac{a(a^2+b^2)^{3/2}}{2(ga^3-fa^2b+eab^2-cb^3)}.$$

So, curvature at the origin $=1/\rho$

$$=\frac{2(ga^3-fa^2b+eab^2-cb^3)}{a(a^2+b^2)^{3/2}}.$$

Again, considering the tangent $y=0$

$$\rho=\underset{x\to 0}{Lt}\ \frac{x^2}{2y}=\underset{x\to 0}{Lt}\ \frac{x^2(ax+by)}{2(cx^3+ex^2y+fxy^2+gy^3)}$$

$$=\underset{x\to 0}{Lt}\ \frac{a+b\left(\frac{y}{x}\right)}{2\left\{c+e\left(\frac{y}{x}\right)+f\left(\frac{y}{x}\right)^2+g\left(\frac{y}{x}\right)^3\right\}}$$

$$=\frac{a}{2c}\qquad\because\ \frac{y}{x}=0$$

So, curvature at the origin

$$=\frac{2c}{a}.$$

10.8. Radius of curvature for polar curves. (C. H. 1962)

Let $P(r, \theta)$ be any point on the curve $r=f(\theta)$, with respect to O as pole and OX as the fixed line, so that $OP=r$ and $\angle POX=\theta$. Let the tangent at P make an angle ψ with OX. Let ϕ be the angle between the radius vector OP and the tangent at P.

Then $\psi=\theta+\phi$

$$\therefore\ \frac{d\psi}{ds}=\frac{d\theta}{ds}+\frac{d\phi}{ds}$$

$$=\frac{d\theta}{ds}+\frac{d\phi}{d\theta}\cdot\frac{d\theta}{ds}$$

$$=\frac{d\theta}{ds}\left(1+\frac{d\phi}{d\theta}\right). \qquad \cdots \qquad (1)$$

Also $\quad \dfrac{ds}{d\theta}=\sqrt{\left\{r^2+\left(\dfrac{dr}{d\theta}\right)^2\right\}} \quad \cdots \quad (2)$

taking positive sign before the radical.

And $\quad \tan\phi=r\dfrac{d\theta}{dr}$

$$=\frac{r}{\dfrac{dr}{d\theta}}=\frac{r}{r_1}$$

$\therefore \quad \sec^2\phi\dfrac{d\phi}{d\theta}=\dfrac{r_1{}^2-r_2r}{r_1{}^2}$

or, $\quad \dfrac{d\phi}{d\theta}=\dfrac{r_1{}^2-r_2r}{r_1{}^2(1+\tan^2\phi)}$

$$=\frac{r_1{}^2-rr_2}{r_1{}^2+r^2} \qquad [\because \ r=r_1\tan\phi.] \quad \cdots \quad (3)$$

Putting (2) and (3) in (1) we get

$$\frac{d\psi}{ds}=\frac{1}{\sqrt{r^2+r_1{}^2}}\left\{1+\frac{r_1{}^2-rr_2}{r^2+r_1{}^2}\right\}$$

$$=\frac{r^2+2r_1{}^2-rr_2}{(r^2+r_1{}^2)^{3/2}} \quad \cdots \quad (4)$$

$\therefore \quad \rho=\dfrac{ds}{d\psi}$

$$=\frac{\left\{r^2+\left(\dfrac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2+2\left(\dfrac{dr}{d\theta}\right)^2-r\dfrac{d^2r}{d\theta^2}}.$$

Cor. At a point of inflexion curvature is zero. So from (4) at a point of inflexion on a polar curve $r^2+2r_1{}^2-rr_2=0$.

Illustrative Examples :

Ex. 1. Show that the radius of curvature of the cardioide $r=a(1+\cos\theta)$ is $\frac{2}{3}\sqrt{2ar}$ (C. H. 1964, 65 old)

$\because \quad r_1=-a\sin\theta,\ r_2=-a\cos\theta.$

$$\rho=\frac{(r^2+r_1{}^2)^{3/2}}{r^2+2r_1{}^2-rr_2}$$

$$=\frac{\{a^2(1+\cos\theta)^2+a^2\sin^2\theta\}^{3/2}}{a^2(1+\cos\theta)^2+2a^2\sin^2\theta+a^2(1+\cos\theta)\cos\theta}$$

$$=\frac{a^3(2+2\cos\theta)^{3/2}}{a^2\{1+\cos^2\theta+2\cos\theta+2\sin^2\theta+\cos\theta+\cos^2\theta\}}$$

$$=\frac{a.2^{3/2}(1+\cos\theta)^{3/2}}{3(1+\cos\theta)}$$

$$=\frac{a\,2\sqrt{2}}{3}(1+\cos\theta)^{\frac{1}{2}}$$

$$=\frac{a\,2\sqrt{2}}{3}\left(\frac{r}{a}\right)^{\frac{1}{2}}$$

$$=\frac{2}{3}\sqrt{2ar}.$$

Ex. 2. Prove that the radius of curvature of a logarithmic spiral is proportional to the radius vector.

(C. H. 1963)

The polar equation of a logarithmic spiral is

$$r=ae^{\theta\cot\alpha}$$

$$\therefore\quad r_1=ae^{\theta\cot\alpha}.\cot\alpha$$

$$r_2=r\cot^2\alpha.$$

$$\therefore\quad \rho=\frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}.$$

$$=\frac{(r^2+r^2\cot^2\alpha)^{3/2}}{r^2+2r^2\cot^2\alpha-r^2\cot^2\alpha}$$

$$=\frac{r^3(1+\cot^2\alpha)^{3/2}}{r^2(1+\cot^2\alpha)}.$$

$$=r(1+\cot^2\alpha)^{\frac{1}{2}}=r\operatorname{cosec}\alpha$$

$$\therefore\quad \rho\propto r.$$

10.9. Radius of curvature for Pedal curves. (C H. 1968, '72)

Let p be the length of the perpendicular from the pole O on the tangent at any point $P(r,\theta)$ on the curve $r=f(\theta)$.

Let the tangent at P make an angle ψ with OX and ϕ be the angle between the radius vector and the tangent at P.

Then, we have $\sin\phi = r\dfrac{d\theta}{ds}$

and $\cos\phi = \dfrac{dr}{ds}$.

Also $p = r\sin\phi$

$$\therefore \quad \frac{dp}{dr} = \sin\phi + r\cos\phi\frac{d\phi}{dr}$$

$$= r\frac{d\theta}{ds} + r\frac{dr}{ds}\cdot\frac{d\phi}{dr}$$

$$= r\frac{d}{ds}(\theta+\phi)$$

$$= r\frac{d\psi}{ds} \qquad [\because \quad \theta+\phi=\psi]$$

$$\Rightarrow \frac{ds}{d\psi} = r\frac{dr}{dp}.$$

Hence, $\rho = r\dfrac{dr}{dp}$.

Illustrative Examples :

Ex. 1. Find ρ for the curve

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}. \qquad \text{(C. H. 1968)}$$

Differentiating the given curve w. r. to p.

$$-\frac{2}{p^3} = -\frac{2r}{a^2b^2}\frac{dr}{dp}.$$

or, $r\dfrac{dr}{dp} = \dfrac{a^2b^2}{p^3}$

Hence $\rho = r\dfrac{dr}{dp} = \dfrac{a^2b^2}{p^3}$.

Ex. 2. Find ρ for the curve $r^m = a^m \cos m\theta$.

$$r^m = a^m\cos m\theta$$

$$\therefore \quad m\log r = m\log a + \log\cos m\theta.$$

Differentiating w. r. to θ

$$\frac{m}{r}\frac{dr}{d\theta}=-\frac{m \sin m\theta}{\cos m\theta}$$

$$\text{or,} \quad \cot\phi=-\tan m\theta$$

$$=\cot\left(\frac{\pi}{2}+m\theta\right)$$

$$\therefore \quad \phi=\frac{\pi}{2}+m\theta.$$

Now $p=r\sin\phi=r\sin\left(\frac{\pi}{2}+m\theta\right)$

$$=r\cos m\theta$$

$$=r.\frac{r^m}{a^m}$$

or, $pa^m=r^{m+1}$ is the pedal equation to the curve.

Differentiating w. r. to r

$$\frac{dp}{dr}=\frac{1}{a^m}(m+1)r^m$$

$$\therefore \quad \rho=r\frac{dr}{dp}=r.\frac{a^m}{(m+1)r^m}=\frac{1}{m+1}.\frac{a^m}{r^{m-1}}.$$

10.10. Radius of Curvature for Tangential Polar Equations. (C. H. 1973)

Let $p=f(\psi)$ be the equation of the curve.

If ϕ be the angle between the radius vector and the tangent at the point of contact and p be the length of the perpendicular from the pole on the tangent then

$$p=r\sin\phi \quad \cdots \quad \cdots \quad (1)$$

$$\cos\phi=\frac{dr}{ds} \quad \cdots \ (2)$$

and $$\rho=r\frac{dr}{dp}. \quad \cdots \ (3)$$

Now $\dfrac{dp}{d\psi}=\dfrac{dp}{dr}\cdot\dfrac{dr}{ds}\cdot\dfrac{ds}{d\psi}$

$=\dfrac{dp}{dr}\cdot\cos\phi\cdot\rho$, from (2)

$=\dfrac{dp}{dr}\cdot\cos\phi\cdot r\dfrac{dr}{dp}$, from (3)

$=r\cos\phi.$

Now $r^2=r^2(\sin^2\phi+\cos^2\phi)$

$=p^2+\left(\dfrac{dp}{d\psi}\right)^2$

$\therefore$ Differentiating w. r. to p.

$$2r\frac{dr}{dp}=2p+2\frac{dp}{d\psi}\,\frac{d^2p}{d\psi^2}\cdot\frac{d\psi}{dp}.$$

or, $$\rho=p+\frac{d^2p}{d\psi^2}.$$

Ex. Find the radius of curvature for the ellipse

$$\mathbf{p^2=a^2\cos^2\psi+b^2\sin^2\psi}$$

Differentiating w. r. to ψ

$$2p\frac{dp}{d\psi}=(b^2-a^2)\sin 2\psi$$

or, $$\frac{dp}{d\psi}=\frac{(b^2-a^2)\sin 2\psi}{2p}.$$

Differentiating again

$$\frac{d^2p}{d\psi^2}=\frac{4p(b^2-a^2)\cos 2\psi-2\dfrac{dp}{d\psi}(b^2-a^2)\sin 2\psi}{4p^2}$$

$$=\frac{4p(b^2-a^2)\cos 2\psi-\dfrac{(b^2-a^2)^2\sin^2 2\psi}{p}}{4p^2}$$

$$\therefore\ p+\frac{d^2p}{d\psi^2}=\frac{4p^4+4p^2(b^2-a^2)\cos 2\psi-(b^2-a^2)^2\sin^2 2\psi}{4p^3}$$

$$=\frac{4p^4+4p^2(b^2-a^2)\cos 2\psi+(b^2-a^2)^2\cos^2 2\psi-(b^2-a^2)^2}{4p^3}$$

$$=\frac{\{2p^2+(b^2-a^2)\cos 2\psi\}^2-(b^2-a^2)^2}{4p^3}$$

$$=\frac{\{2a^2\cos^2\psi+2b^2\sin^2\psi+(b^2-a^2)(\cos^2\psi-\sin^2\psi)\}^2-(b^2-a^2)^2}{4p^3}$$

$$=\frac{(a^2+b^2)^2-(b^2-a^2)^2}{4p^3}$$

$$=\frac{4a^2b^2}{4p^3}=\frac{a^2b^2}{p^3}$$

$$\therefore \quad \rho=p+\frac{d^2p}{d\psi^2}=\frac{a^2b^2}{p^3}.$$

10.11. The centre of curvature

If ρ be the radius of curvature of point P of a curve and if a point C is taken along the positive direction of the normal at P at a distance ρ from P, then the point C is called the centre of curvature of the curve.

The circle of curvature

If C be the centre of curvature with respect to a point P on a curve and if a circle be drawn with centre at C and radius equal to the radius of curvature at P, then this circle is called the circle of curvature. Clearly, the circle of curvature will touch the circle at P and its curvature will be the same as that of the curve at P.

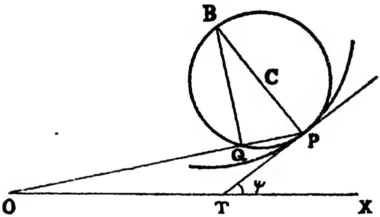
Chord of curvature

If a circle of curvature at a point P on a curve be drawn and if a chord to the circle of curvature be drawn through P in any given direction, then that chord is called the chord of curvature of P in that direction.

Chord of curvature through the pole.

Let ρ be the radius of curvature of a point P of a curve. Let PQ be a chord of the circle of curvature through P in the direction of the pole. Let C be the centre of the circle of curvature and let the join of PC meet this circle at B so

that $PB=2\rho$. Let the tangent at P make an angle ψ with OX.



Then chord $PQ=PB \sin PBQ$

$$=2\rho \sin QPT$$

$$=2\rho \sin \phi$$

$$=2r\frac{dr}{dp}\cdot\frac{p}{r}. \qquad \left[\because\ p=r\sin\phi \text{ and } \rho=r\frac{dr}{dp}\right].$$

$$=2p\frac{dr}{dp}.$$

Thus, the chord of curvature of a point on a curve in the direction of the pole is

$$2p\frac{dr}{dp},$$

and which can be easily obtained if the pedal equation to the curve be given.

Note. If the chord PQ instead of passing through the pole be parallel to x-axis, then

$$\angle OPT=\psi$$

i.e., $\alpha=\psi$, [Denoting $\angle OPT=\alpha$]

Then $PQ=PB \sin PBQ$

$$=2\rho \sin \alpha$$

$$=2r\frac{dr}{dp} \sin \psi.$$

Similarly, if the chord PQ be parallel to y-axis then

chord $PQ = 2\rho \sin\left(\frac{\pi}{2} - \alpha\right)$

$$= 2\rho \cos \alpha$$

$$= 2r\frac{dr}{dp} \cos \psi$$

Cor. The chord of curvature through the pole for the curve $p = f(r)$ is given by $2f(r)/f'(r)$. (C. H. 1972)

From the given equation $\frac{dp}{dr} = f'(r)$

$\therefore$ Chord of curvature through the pole

$$= 2p\frac{dr}{dp} = 2f(r)/f'(r)$$

Ex. For the curve $y = a \log \sec\left(\frac{x}{a}\right)$, find the chord of curvature parallel to x and y axes.

re $$\frac{dy}{dx} = \frac{a}{\sec \frac{x}{a}} \cdot \sec \frac{x}{a} \cdot \tan \frac{x}{a} \cdot \frac{1}{a}$$

$$= \tan \frac{x}{a}.$$

$$\therefore \quad \tan \psi = \tan \frac{x}{a}$$

$$\Rightarrow \psi = \frac{x}{a}.$$

$$\therefore \quad \rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1 + \tan^2 \frac{x}{a}\right)^{3/2}}{\frac{1}{a}\sec^2 \frac{x}{a}}$$

$$= a \sec \frac{x}{a}.$$

$\therefore$ Chord of curvature parallel to x axis

$$=2\rho \sin \psi$$

$$=2a \sec \frac{x}{a} . \sin \frac{x}{a} .$$

$$=2a \tan \frac{x}{a} .$$

Also chord of curvature parallel to y axis

$$=2\rho \cos \psi$$

$$=2a \sec \frac{x}{a} \cos \frac{x}{a} .$$

$$=2a \text{ (constant).}$$

10.12. Cartesian co-ordinates of the centre of curvature for any point P(x, y) of the curve y = f(x)

Let ρ be the radius of curvature at any point $P(x, y)$ of the curve $y=f(x)$

$$\text{Then, } \rho=\frac{(1+y_1{}^2)^{3/2}}{y_2}, \; y_2 \neq 0. \quad \cdots \quad \cdots \quad (1)$$

Let the tangent at P make an angle ψ with the positive direction of the x axis.

$$\text{Then, } \sin \psi=\frac{y_1}{\sqrt{1+y_1{}^2}}, \; \cos \psi=\frac{1}{\sqrt{1+y_1{}^2}} \quad \cdots \quad (2)$$

The normal at P makes an angle $\left(\frac{\pi}{2}+\psi\right)$ with the positive direction of the x axis

$\therefore$ The equation to the normal at P is

$$\frac{X-x}{\cos\left(\frac{\pi}{2}+\psi\right)}=\frac{Y-y}{\sin\left(\frac{\pi}{2}+\psi\right)}=r$$

$$\text{or, } \frac{X-x}{-\sin \psi}=\frac{Y-y}{\cos \psi}=r$$

where X, Y are the current co-ordinates at any point on the normal at a variable distance r from $P(x, y)$.

$$\therefore \quad X = x - r \sin\psi$$
$$Y = y + r \cos\psi.$$

If now (X, Y) coincides with the centre of curvature, then r becomes equal to ρ.

Hence, if (X, Y) be the co-ordinates of the centre of curvature, then

$$X = x - \rho \sin\psi$$
$$= x - \frac{(1+y_1^2)^{3/2}}{y_2} \cdot \frac{y_1}{\sqrt{1+y_1^2}}$$
$$= x - \frac{y_1(1+y_1^2)}{y_2} \quad \dots \quad \dots \quad \dots \quad (3)$$

And $Y = y + \rho \cos\psi$

$$= y + \frac{(1+y_1^2)^{3/2}}{y_2} \cdot \frac{1}{\sqrt{1+y_1^2}}$$
$$= y + \frac{1+y_1^2}{y_2}. \quad \dots \quad \dots \quad \dots \quad (4)$$

Cor. Equation of the circle of curvature is

$$(x-X)^2 + (y-Y)^2 = \rho^2$$

where X and Y are determined by (3) and (4) and ρ is determined by (1).

Illustrative Examples :

Ex. 1. Show that the centre of curvature (α, β) for the curve $a^2y = x^3$ is given by

$$\alpha = \frac{x}{2}\left(1 - \frac{9x^4}{a^4}\right), \quad \beta = \frac{5}{2}\frac{x^3}{a^2} + \frac{a^2}{6x}. \qquad \text{(C. H. 1971)}$$

The given curve is $y = \dfrac{x^3}{a^2}$.

$$\therefore \quad y_1 = \frac{3x^2}{a^2}, \; y_2 = \frac{6x}{a^2}.$$

Now $\alpha=x-\dfrac{y_1(1+y_1^2)}{y_2}$

$$=x-\frac{\dfrac{3x^2}{a^2}\left(1+\dfrac{9x^4}{a^4}\right)}{6x/a^2}$$

$$=\frac{6x^2-3x^2-27x^6/a^4}{6x}$$

$$=\frac{1}{6}\left(3x-27x^5/a^4\right)$$

$$=\frac{x}{2}\left(1-\frac{9x^4}{a^4}\right).$$

And $\beta=y+\dfrac{1}{y_2}\left(1+y_1^2\right)$

$$=y+\frac{a^2}{6x}\left(1+\frac{9x^4}{a^4}\right).$$

$$=y+\frac{a^2}{6x}+\frac{3x^3}{2a^2}$$

$$=\frac{x^3}{a^2}+\frac{a^2}{6x}+\frac{3}{2}\frac{x^3}{a^2}$$

$$=\frac{5}{2}\frac{x^3}{a^2}+\frac{a^2}{6x}.$$

Ex. 2. Find the circle of curvature of the parabola $y^2=4x$ at the ends of its latus rectum.

Here $2yy_1=4 \qquad \therefore \quad y_1=\dfrac{2}{y}$

$$y_2=-\frac{2}{y^2}\cdot y_1=-\frac{4}{y^3}.$$

Let $L(1, 2)$ be the upper extremity of the latus rectum then at $L(1, 2)$, $y_1=1$, $y_2=-\dfrac{1}{2}$

If (X, Y) be the co-ordinate of the centre of curvature hen $X=x-\dfrac{y_1(1+y_1^2)}{y_2}$

$$1-\frac{1(1+1)}{-\frac{1}{2}}=5$$

$$Y=y+\frac{1+y_1^2}{y_2}$$

$$=2+\frac{1+1}{-\frac{1}{2}}=-2.$$

Thus, the circle of curvature passes through $L(1, 2)$ and has the centre at $(5, -2)$

Hence, equation to this circle of curvature is

$$(x-5)^2+(y+2)^2=(1-5)^2+(2+2)^2$$

$$=16+16=32$$

or, $x^2+y^2-10x+4y-3=0.$

Similarly, equation to the circle of curvature passing through $L'(1, -2)$ is

$$x^2+y^2-10x-4y-3=0.$$

10.13. Evolute : The locus of the centres of curvature of a curve is called the evolute of the curve, and the curve is called the involute of the evolute.

Properties of an Evolute :

(*a*) **The normals to a curve are the tangents to its evolute.**

Let ρ be the radius of curvature of a point $P(x, y)$ on a curve and let (X, Y) be the co ordinates of the centre of curvature with respect to P on the curve. Then

$$X=x-\rho\sin\psi \qquad \cdots \qquad \cdots \qquad (1)$$

$$Y=y+\rho\cos\psi \qquad \cdots \qquad \cdots \qquad (2)$$

Differentiating w. r. to x

$$\frac{dX}{dx}=1-\rho\cos\psi\frac{d\psi}{dx}-\sin\psi\frac{d\rho}{dx}.$$

$$=1-\frac{ds}{d\psi}\cdot\frac{dx}{ds}\cdot\frac{d\psi}{dx}-\sin\psi\frac{d\rho}{dx}$$

$$=-\sin\psi\frac{d\rho}{dx}. \qquad \cdots \qquad \cdots \qquad (3)$$

$$\frac{dY}{dx}=\frac{dy}{dx}-\rho\sin\psi\frac{d\psi}{dx}+\cos\psi\frac{d\rho}{dx}$$

$$=\frac{dy}{dx}-\frac{ds}{d\psi}\cdot\frac{dy}{ds}\cdot\frac{d\psi}{dx}+\cos\psi\frac{d\rho}{dx}$$

$$=\cos\psi\frac{d\rho}{dx}. \quad \ldots \quad \ldots \quad (4)$$

$$\therefore \quad \frac{dY}{dX}=-\cot\psi.$$

$$=\tan\left(\frac{\pi}{2}+\psi\right)$$

= slope of the normal at P on the original curve.

But $\frac{dY}{dX}$ is the slope of the tangent to the evolute at P' corresponding to the point P on the original curve.

Hence, the normals to a curve are the tangents to its evolute.

(*b*) **The difference between the radii of curvatures at two points of a curve is equal to the length of the arc of the evolute between the two corresponding points.**

Squaring (3), (4) and adding

$$\left(\frac{dX}{dx}\right)^2+\left(\frac{dY}{dx}\right)^2=\left(\frac{d\rho}{dx}\right)^2$$

$$\text{or,} \quad \left(\frac{ds}{dx}\right)^2=\left(\frac{d\rho}{dx}\right)^2$$

$$\therefore \quad \frac{ds}{dx}=\frac{d\rho}{dx}.$$

$\therefore \quad s=\rho+c$ where c is a constant.

Let s_1 and s_2 be the arc distances of two points Q_1 and Q_2 on the evolute corresponding to the two point P_1 and P_2 on the original curve of radii of curvature ρ_1 and ρ_2 respectively. Then

$$s_1=\rho_1+c$$

$$s_2=\rho_2+c$$

$$\therefore \quad s_2-s_1=\rho_2-\rho_1.$$

Hence, the theorem.

Illustrative Examples :

Ex 1. Find the evolute of the parabola $y^2=4ax$

From $y^2=4ax$

$$y_1=\sqrt{\frac{a}{x}} \text{ and } y_2=-\frac{\sqrt{a}}{2x^{3/2}}.$$

If (X, Y) be the centre of curvature

then $$X=x-\frac{y_1(1+y_1^2)}{y_2}=x+\frac{\sqrt{\frac{a}{x}}\left(1+\frac{a}{x}\right)}{\frac{\sqrt{a}}{2x^{3/2}}}$$

$$=x+\frac{2x^{3/2}}{\sqrt{x}}\left(\frac{a+x}{x}\right)=x+2(a+x)$$

$$=3x+2a. \qquad \cdots \qquad (1)$$

$$Y=y+\frac{1+y_1^2}{y_2}=y-\frac{1+\frac{a}{x}}{\sqrt{a}/2x^{3/2}}$$

$$=2\sqrt{ax}-\frac{2\sqrt{x}(x+a)}{\sqrt{a}}=-\frac{2x^{3/2}}{\sqrt{a}}. \qquad \cdots \qquad (2)$$

Now eliminating (x, y) from (1) and (2) the required evolute is obtained.

From (1), $x=\dfrac{X-2a}{3}$

and from (2), $Y=-\dfrac{2}{\sqrt{a}}\left(\dfrac{X-2a}{3}\right)^{\frac{3}{2}}$

Squaring, $27aY^2=4(X-2a)^3$

Hence, writing x, y for X, Y, the required evolute is

$$27ay^2=4(x-2a)^3.$$

Ex. 2. Find the length of the arc of the evolute of the parabola $y^2=4ax$ intercepted by the parabola.

The evolute is

$$27ay^2=4(x-2a)^3.$$

The x co-ordinates of the point of intersection of this evolute and the parabola $y^2=4ax$ is obtained from

$$27a(4ax)=4(x^3-6x^2a+12xa^2-8a^3)$$

i.e., from $x^3-6ax^2-15a^2x-8a^3=0$.

Solving, we get $x=8a$ is the only positive root.

$\therefore \quad y=\pm 4\sqrt{2}a.$

$\therefore$ The points of intersection are

$(8a, 4\sqrt{2}a)$ and $(8a, -4\sqrt{2}a)$.

Now consider $P'(8a, 4\sqrt{2}a)$.

Let $(8a, 4\sqrt{2}a)$ be the centre of curvature corresponding to the point $P(x, y)$ on the parabola.

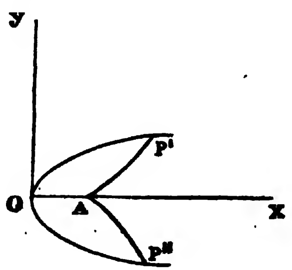
But if (X, Y) be the centre of curvature for any point (x, y) on the curve, then

$$X=3x+2a, \quad Y=-y^3/4a^2$$

$$\therefore \quad 8a=3x+2a \text{ and } 4\sqrt{2}a=-\frac{y^3}{4a^2}$$

so $\quad x=2a, y=-2\sqrt{2}a$

Hence, $P'(8a, 4\sqrt{2}a)$ is the centre of curvature corresponding to $P(2a, -2\sqrt{2}a)$ on the parabola.



If ρ_2 be the radius of curvature at P then

$$\rho_2=\frac{(1+y_1^2)^{3/2}}{y_2} \quad \left[\because \quad 2yy_1=4a\right.$$

$$\therefore \quad y_1=\frac{2a}{y}=-\frac{2a}{2\sqrt{2a}}=-\frac{1}{\sqrt{2}}.$$

$$y_2=-\frac{2a}{y^2}\cdot y_1=-\frac{2a}{8a^2}\left(-\frac{1}{\sqrt{2}}\right)$$

$$=\frac{1}{4\sqrt{2a}}\ ]$$

$$=\frac{(1+\frac{1}{2})^{3/2}}{\frac{1}{4\sqrt{2a}}}$$

$$=\frac{3^{3/2}.\,4\sqrt{2a}}{2^{3/2}}$$

$$=2a\sqrt{27}=6\sqrt{3}a$$

Also ρ_1 at $0(0, 0)$ is $2a$.

Let A be the centre of curvature corresponding to the point $0(0, 0)$ on the parabola.

$\therefore$ Length of the arc AP' on the evolute

$$=\rho_2-\rho_1=6\sqrt{3}a-2a$$

$$=2a(3\sqrt{3}-1)$$

$\therefore$ Whole length of the evolute

$$=4a(3\sqrt{3}-1).$$

Exercise 10

1. In any curve show that

(a) $\rho=\left\{\left(\frac{dx}{d\psi}\right)^2+\left(\frac{dy}{d\psi}\right)^2\right\}^{1/2}.$

(b) $\frac{1}{\rho^2}=\frac{d^2x}{ds^2}\,\frac{d^3y}{ds^3}-\frac{d^2y}{ds^2}\,\frac{d^3x}{ds^3}.$

2. In any curve if $r=1/u$, show that

(a) $\rho=\frac{(u^2+u_1{}^2)^{3/2}}{u^3(u+u_2)}\ ,u^3(u+u_2)\neq 0).$

(b) $\rho=r\frac{d\theta}{dr}\Big/\left\{r\left(\frac{d\theta}{ds}\right)^2-\frac{d^2r}{ds^2}\right\}.$

3. Show that curvature at the point (s, ψ) of the curve

$s=a\{\sec\psi\tan\psi+\log(\sec\psi+\tan\psi)\}$ is $\frac{1}{2a}\cos^2\psi.$

4. Show that the radius of curvature of the curves

(a) $y=a\log\sec\left(\frac{x}{a}\right)$ at the point (x, y) is $a\sec\left(\frac{x}{a}\right).$

(b) $y=\frac{1}{2}a\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ at the point (x, y) is y^2/a.

(c) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ at the point (x, y) is $3\,(axy)^{\frac{1}{3}}$.

5. If ρ be the radius of curvature at any point P of the parabola $y^2=4ax$, show that

$$\rho^2 \propto (SP)^3.$$

6. Show that the radius of curvature of the curve $x^2y=a(x^2+y^2)$ at the point $(-2a, 2a)$ is $-2a$.

7. Show that the radius of curvature at $(0, 0)$ of the curve $y^2(a-x)=x^2(a+x)$ is $4\sqrt{2}a$.

8. For the curve $x=ae^{\theta}(\sin\theta-\cos\theta)$, $y=ae^{\theta}(\sin\theta+\cos\theta)$ show that the radius of curvature at a point is equal to the distance from the origin to the tangent at the point.

9. Show that for the curve

$$r^n=a^n\cos n\theta,$$

the radius of curvature $\rho=\dfrac{a^n}{(n+1)r^{n-1}}$.

10. For the cardioide $r=a(1+\cos\theta)$, show that the radius of curvature at a point where the tangent is parallel to the initial line is $\frac{2}{3}\sqrt{3}a$.

Also prove that at any point of this curve ρ^2/r is constant.

11. Prove that the radius of curvature at a point (p, r) of the curve

$$p^2(a^2+b^2-r^2)=a^2b^2$$

is a^2b^2/p^3.

12. Show that the radius of curvature at any point ψ of curve $p=a\sin\psi\cos\psi$ is $3p$.

13. If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord passing through the pole of the cardioide $r=a(1+\cos\theta)$, show that

$$\rho_1^2+\rho_2^2=16\,a^2/9.$$

14. Show that the evolute of the ellipse $x^2/a^2+y^2/b^2=1$ is $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$.

15. Prove that the evolute of the hyperbola $xy=\frac{1}{2}c^2$ is $(x+y)^{\frac{2}{3}}-(x-y)^{\frac{2}{3}}=2c^{\frac{2}{3}}$.

16. Show that for the curve $y=a\log\sec\left(\dfrac{x}{a}\right)$, the chord of curvature parallel to y axis is of constant length.

17. Show that the circle of curvature for the parabola

$\sqrt{x}+\sqrt{y}=\sqrt{a}$ at a point $\left(\frac{a}{4}, \frac{a}{4}\right)$ is

$$\left(x-\frac{3}{4}a\right)^2+\left(y-\frac{3}{4}a\right)^2=\frac{1}{2}a^2$$

18. Show that the chord of curvature of the cardioide $r=a(1+\cos\theta)$, through the pole is $\frac{4}{3}r$.

19. Show that the chord of curvature through the pole of the curve $p=ae^{br}$ is of constant length.

20. Let ρ_1 and ρ_2 be the radii of curvature at the corresponding points of a cycloid and its evolute. Prove that $\rho_1^2+\rho_2^2=$constant.

21. Let C_x and C_y denote the chords of curvature at any point of the curve $y=ae^{\frac{x}{a}}$, parallel to the axis of x and y respectively. Prove that

$$\frac{1}{(C_x)^2}+\frac{1}{(C_y)^2}=\frac{1}{2aC_x}.$$

22. Let C_x and C_y denote the chords of curvature at any point of the catenary $y=c\cosh\left(\frac{x}{c}\right)$, prove that

$$C_x^2+C_y^2=\frac{cy^4}{4c^2}.$$

23. If r_1 be the distance between the pole and the centre of curvature corresponding to any point on the curve $r=f(\theta)$, prove that

$$r_1^2=r^2+\rho^2-2p\rho$$

where ρ and p have their usual meanings.

24. If (α, β) be the co-ordinates of the centre of curvature of the parabola $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at any point (x, y) then prove that

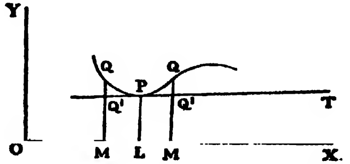
$$\alpha+\beta=3(x+y).$$

CHAPTER XI

POINTS OF INFLECTION AND SINGULARITIES

11.1. Concavity :

If a point P be taken on a certain portion of a curve however small and a tangent at P be drawn so that portion of the curve lies wholly above the tangent (i.e., the positive direction of the y axis) then the curve is said to be concave upwards or convex downwards at P.



Let PT be a tangent at P to the portion of the curve $y=f(x)$ at $x=c$ so that the portion of the curve around P however small it may lie enterely above the tangent. Then according to definition, the curve is concave upwards at P.

Let the co-ordinates at c and $c+h$ where h is positive or negative be drawn, h being very small.

Then $f(c)=PL$ and $f(c+h)=QM$.

Equation to the tangent at $P(c, f(c))$ is

$$y-f(c)=f'(c)(x-c).$$

When $x=c+h$

$$y=f(c)+hf'(c)$$

$$\Rightarrow Q'M=f(c)+hf'(c)$$

$$\therefore \quad QQ'=QM-Q'M=f(c+h)-f(c)-hf'(c).$$

Since, the curve is concave upwards at P.

$$QM>Q'M$$

$$\Rightarrow QM-Q'M>0$$

$$\Rightarrow Q'Q>0$$

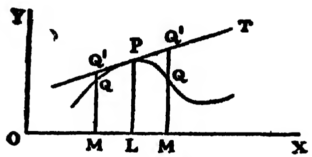
$$\Rightarrow f(c+h)-f(c)-hf'(c)>0.$$

Hence the curve $y=f(x)$ is concave upwards at $x=c$ if $f(c+h)>f(c)+hf'(c)$ when h is sufficiently small, positive or negative.

11.2. Convexity :

If a point P be taken on a certain portion of a curve however small it may be and a tangent at P be drawn so that the portion of the curve lies entirely below the curve towards the negative of the y axis, then curve is said to be convex upwards or concave downwards at P.

Let PT be a tangent to the portion of the curve $y=f(x)$ at $x=c$ so that the portion of the curve however small it may lie entirely below the tangent at $x=c$.



Then according to definition, the curve is convex upwards at $x=c$.

Then $f(c)=PL$ and $f(c+h)$ $=QM$ where h is very small positive or negative.

Equation to the tangent at $P\{(c, f(c)\}$ is

$$y-f(c)=f'(c)(x-c)$$

When $x=c+h$

$$y=f(c)+hf'(c)$$

$$\Rightarrow Q'M=f(c)+hf'(c).$$

Since, the curve is convex upwards at $x=c$

$$QM<Q'M$$

$$\Rightarrow QM-Q'M<0$$

$$\Rightarrow f(c+h)-f(c)-hf'(c)<0$$

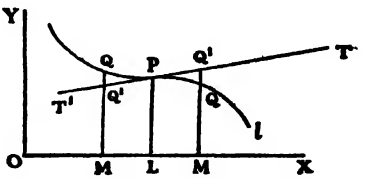
Hence, the curve $y=f(x)$ is convex upwards at $x=c$ if $f(c+h)<f(c)+hf'(c)$.

11.3. Inflextion ;

If a point P be taken to a certain portion of curve however small it may be and a tangent be drawn at P,

then P is the point of inflexion if the tangent crosses the curve at P.

Let a tangent at P to the portion of the curve $y=f(x)$ be drawn at $x=c$ so that the tangent crosses the curve at P. In the diagram, the portion Pl of the curve lies below the tangent and the portion Pl' lies above the tangent. So according to definition P is the point of inflexion at $x=c$.



Equation to the tangent at $P\{c, f(c)\}$ is

$$y-f(c)=f'(c)(x-c)$$

when $\qquad x=c+h$

$$y=f(c)+hf'(c)$$

$\Rightarrow Q'M=f(c)+hf'(c)$ where h is positive or negative.

But $\quad QM=f(c+h)$ when h is positive or negative.

So if h be positive

$$QM-Q'M<0$$

$$\Rightarrow f(c+h)<f(c)+hf'(c)$$

And if h be negative

$$QM-Q'M>0$$

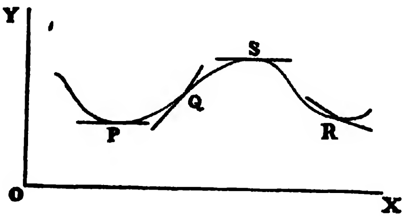
$$\Rightarrow f(c+h)>f(c)+hf'(c)$$

Hence $x=c$ is the point of inflexion of the curve $y=f(x)$ if $f(c)+hf'(c)$ changes sign with the change of sign of h.

11.4. Alternative definition of Inflection.

From the adjoined diagram, we find the curve is concave upwards and downward at P and S respectively. Q and R are the points of inflexion. The curve is concave upwards at every point between P and Q and concave downwards at every point between Q and R. So the curve changes from

concavity to convexity or vice versa as it passes through a point of inflexion. Hence, a point P on the curve may be



said to be a point of inflexion if the curve be such that as it crosses P, it changes its bending from concavity to convexity or vice versa

11.5. *Theorem :*

A curve $y=f(x)$ at $x=c$ is concave upwards if $f''(c)>0$ and concave downwards if $f''(c)<0$. The point P at $x=c$ is a point of inflexion if $f''(c)=0$ and $f'''(c)\neq 0$.

Proof: By Taylor's Theorem with remainder after two terms

$$f(c+h)=f(c)+hf'(c)+\frac{h^2}{\lfloor 2}f''(c+\theta h),\ 0<\theta<1.$$

or, $$f(c+h)-f(c)-hf'(c)=\frac{h^2}{\lfloor 2}f''(c+\theta h) \tag{1}$$

Case I. Let $f''(c)>0$.

As $f''(x)$ is positive at $x=c$, there exists an interval around c such that for every x of this interval $f''(x)$ has the same sign as $f''(c)$ which is positive by hypothesis. Hence, $f''(x)$ is positive for every x of this interval. If $c+h$ is a point of this interval, then $c+\theta h$ is also a point of this interval, and so $f(c+\theta h)$ is positive. Also $h^2/\lfloor 2$ is always positive.

Then from (1) we find that if $f''(c)>0$

Then $f(c+h)-f(c)-hf'(c)>0$

$\Rightarrow f(c+h)>f(c)+hf'(c)$, which is the property of concavity upwards at $x=c$.

Hence the curve is concave upwards at $x=c$ if $f''(c)>0$.

Case II. Let $f''(c)<0$.

As $f''(c)$ is negative at $x=c$, there exists an interval around c such that for every x of this interval $f''(x)$ has the same sign as $f''(c)$ which is negative here. So $f''(x)$ is negative for every x of this interval. If we now take a point $c+h$ within this interval then $c+\theta h$ will also lie within this interval and consequently $f(c+\theta h)$ must be negative. But $h^2/\underline{|2}$ is always positive.

Thus, from (1), we find that if $f''(c)<0$ then $f(c+h)-f(c)-hf'(c)<0$

$\Rightarrow f(c+h)<f(c)+hf'(c)$

which is the property of convexity upwards at $x=c$.

Hence, the curve is convex upwards at $x=c$ if $f''(c)<0$.

Case III. Let $f''(c)=0$, but $f'''(c)\neq 0$.

By Taylors Theorem with remainder after three terms

$$f(c+h)=f(c)+hf'(c)+\frac{h^2}{\underline{|2}}f''(c)+\frac{h^3}{\underline{|3}}f'''(c+\theta h),\ 0<\theta<1$$

$$=f(c)+hf'(c)+\frac{h^3}{\underline{|3}}f'''(c+\theta h). \qquad \because\ f''(c)=0.$$

$$or,\quad f(c+h)-f(c)-hf'(c)=\frac{h^3}{\underline{|3}}f'''(c+\theta h). \cdots \quad (2)$$

Now there exists an interval around c such that for every x of this interval $f'''(x)$ has the same sign as $f'''(c)$. If now a point $c+h$ be taken within this interval then $c+\theta h$ will also lie within this interval and accordingly $f'''(c+\theta h)$ will have the same sign as $f'''(c)$. But $h^3/\underline{|3}$ is positive or negative accordidg as h is positive or negative.

Thus $\frac{h^3}{\underline{|3}}f'''(c+\theta h)$ changes sign with the change of sign

in h. So from (2) we find that $f(c+h)-f(c)-hf'(c)$ changes sign with the change in the sign of h. But this is the property of inflexion of the curve at $x=c$.

Hence, the curve has an inflexion at $x=c$ if $f''(c)=0$, but $f'''(c)\neq 0$.

11.6. Generalised Theorem.

Let $f''(c)=f'''(c)=\cdots f^{n-1}(c)=0$, but $f^{n}(c)\neq 0$. Then (i) if n is odd the curve has an inflexion at $x=c$ and (ii) if n is even the curve is concave upwards or downwards according as $f^{n}(c)>0$ or <0.

Proof. By Taylor's Theorem with Lagranges form of remainder after n terms at $x=c$, we have

$$f(c+h)=f(c)+hf'(c)+\frac{h^2}{\lfloor 2}f''(c)+\cdots+\frac{h^{n-1}}{\lfloor n-1}f^{n-1}(c)$$

$$+\frac{h^n}{\lfloor n}f^{n}(c+\theta h) \text{ for } 0<\theta<1$$

or, $$f(c+h)-f(c)-hf'(c)=\frac{h^n}{\lfloor n}f^{n}(c+\theta h) \quad \cdots \quad (1)$$

Now there exists an interval around c such that for every x of this interval $f^{n}(x)$ has the same sign as $f^{n}(c)$. If a point $c+h$ be taken within this interval then $c+\theta h$ will also lie within the same interval and accordingly $f^{n}(c+\theta h)$ will have the same sign as $f^{n}(c)$.

But $h^n/\lfloor n$ changes its sign or keeps the same sign when h changes its sign, according as n is odd or even.

Thus from (1) we find that

$$f(c+h)-f(c)-hf'(c)$$

changes its sign when n is odd and keeps the same sign of $f^{n}(c)$ if n is even.

Hence, the curve has an inflexion at $x=c$ if n be odd. Also it is concave upwards or downwards at $x=c$ if n be even according as $f^{n}(c)>0$ or <0.

Note : Unlike concavity and convexity at a point of a curve, the position of the point of inflexion of a curve is an inherent property of the curve and so it is independent of the choice of axis. Hence, the position of the point of inflexion will not change if we interchange the axis and accordingly the point of inflexion may also be examined by considering $\frac{d^2x}{dy^2}$.

Ex. 1. Show that $y=x^6$ is concave upwards at the origin.

We have $\frac{dy}{dx}=6x^5$, $\frac{d^2y}{dx^2}=30x^4$

$$\frac{d^3y}{dx^3}=120x^3,\quad \frac{d^4y}{dx^4}=360x^2,\quad \frac{d^5y}{dx^5}=720x$$

and $\frac{d^6y}{dx^6}=720$

$\therefore$ At the origin (0, 0), $\frac{dy}{dx}=\frac{d^2y}{dx^2}=\frac{d^3y}{dx^3}=\frac{d^4y}{dx^4}=\frac{d^5y}{dx^5}=0$

but $\frac{d^6y}{dx^6}\neq 0$ and positive.

Since, even differential co-efficient is positive and $\neq 0$, the given curve is concave upwards at the origin.

Ex. 2. Prove that the points of intersection of the curve $(x+y-a)^3+27axy=0$ with the line $x+y=a$ are points of inflexion on the curve. **(C. H. 1959)**

Solving, the points of intersection are $(a, 0)$ and $(0, a)$.

Differentiating $(x+y-a)^3+27axy=0$, *w. r.* to x

$$3(x+y-a)^2\left\{1+\frac{dy}{dx}\right\}+27a\left\{y+x\frac{dy}{dx}\right\}=0.$$

or, $\{3(x+y-a)^2+27ax\}\frac{dy}{dx}=-3(x+y-a)^2-27ay \quad \cdots (1)$

or, $\frac{dy}{dx}=-\frac{(x+y-a)^2+9ay}{(x+y-a)^2+9ax}=0$ at $(a, 0)$.

Differentiating (1) again

$$\left\{6(x+y-a)^2\left(1+\frac{dy}{dx}\right)+27a\right\}\frac{dy}{dx}+\{3(x+y-a)^2+27ax\}\frac{d^2y}{dx^2}$$
$$+6(x+y-a)\left(1+\frac{dy}{dx}\right)+27a\frac{dy}{dx}=0. \quad \cdots \quad (2)$$

$$\therefore \quad \frac{d^2y}{dx^2}=0 \text{ at } (a, 0)$$

Differentiating (2) again and using $\frac{dy}{dx}=0,\ \frac{d^2y}{dx^2}=0$

$$\{3(x+y-a)^2+27ax\}\frac{d^3y}{dx^3}+6=0$$

or, $$\frac{d^3y}{dx^3}=-\frac{6}{3(x+y-a)^2+27ax}=-\frac{6}{27a^2}\neq 0$$

Since odd differential co-efficient is negative and $\neq 0$, the point $(a, 0)$ is the point of inflexion. Similarly, $(0, a)$ is the point of inflexion.

11.7. Singularities.

The curve of the implicit equation $f(x, y)=0$ some times exhibit pecularities at certain points with regard to its tangents at those points. This is due to the fact that for any real value of x in $f(x, y)=0$, y may have as many real values as the number of degree of the equation. So, the curve may have as many branches as the number of degree of the equation. *A point through which there pass r branches of a curve is called a singular point or a multiple point of the r-th order.*

11.8. Conjugate or Isolated point.

If the coordinates of a point satisfy the equation of the curve but no point in the immediate neighbourhood of that point lies on the curve, then that point is called a conjugate point or an isolated point.

The curve $y^2=x(x+a)^2$ has two branches from the origin lying on the 1st and 4th quadrant respectively. But

$(-a, 0)$ also satisfy the equation but points lying between $(-a, 0)$ and $(0, 0)$ are not lying on the curve. Hence $(-a, 0)$ is a conjugate point of the curve.

Node. A point through which emerges two branches of the curve with two tangents at that point is called a Node.

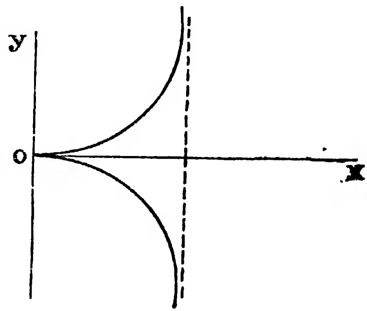
The Folium of Descartes $x^3+y^3=3axy$ has two branches emerging out from the origin lying on the 2nd and 4th quadrant respectively and has two tangents $y=0$ and $x=0$ at the origin. Hence, origin is a Node of the curve.

Cusp. A point through which there pass two branches of a curve with a common tangent at that point is called a cusp.

The Cissoid

$$y^2(a-x)=x^3$$

has two branches emerging out from the origin lying on the 1st and 4th quadrant respectively and have a common tangent $y=0$ at the origin. Hence, origin is a cusp of the curve.



Double point. A point which is either a Node or a Cusp or a conjugate point is called a double point.

11.9. Conditions for which the curve $f(x, y)=0$ attains a multiple point at any point (x, y).

From the given equation the slope of the tangent $\frac{dy}{dx}$ is given by

$$f_x+f_y\frac{dy}{dx}=0 \quad \cdots \quad \cdots \quad (1)$$

At a multiple point the curve must have at least two tangents and accordingly $\frac{dy}{dx}$ must have at least two values real equal or real unequal.

But the equation (1) is an equation of the first degree in $\frac{dy}{dx}$ and so it can be satisfied by more than two values of $\frac{dy}{dx}$, only if $f_x=0$, and $f_y=0$.

So a point (x, y) can be a multiple point of the curve $f(x, y)=0$ if and only if

$$f_x(x, y)=0 \text{ and } f_y(x, y)=0$$

Hence, a point (x, y) will be a multiple point of the curve if it satisfy simultaneously the three equations

$$f(x, y)=0$$
$$f_x(x, y)=0$$
$$\text{and} \quad f_y(x, y)=0.$$

Again, differentiating (1) w. r. to x, we get

$$f_{xx}+f_{xy}\frac{dy}{dx}+\left(f_{yx}+f_{yy}\frac{dy}{dx}\right)\frac{dy}{dx}+f_y\frac{d^2y}{dx^2}=0. \qquad \cdots \quad (2)$$

$\therefore$ At a multiple point, where $f_x=f_y=0$

we get from (2)

$$f_{xx}+2f_{xy}\frac{dy}{dx}+f_{yy}\left(\frac{dy}{dx}\right)^2=0.$$

$$\text{or,} \quad f_y^2\left(\frac{dy}{dx}\right)^2+2f_{xy}\left(\frac{dy}{dx}\right)+f_x^2=0.$$

$$\text{or,} \quad \frac{dy}{dx}=\frac{-2f_{xy}\pm\sqrt{4f_{xy}^2-4f_x^2f_y^2}}{2f_y^2}$$

$$=\frac{-f_{xy}\pm\sqrt{f_{xy}^2-f_x^2f_y^2}}{f_y^2}.$$

So, the slope of the tangent at a multiple point is real unequal, real equal or imaginary according as

$$f_{xy}^2-f_x^2f_y^2>0, \ =0 \text{ or, } <0$$

provided f_x^2, f_{xy}, f_y^2 are not all zero.

Hence, if a point (x, y) simultaneously satisfy the three equations,

$$f(x, y)=0, \ f_x(x, y)=0, \ f_y(x, y)=0$$

then the point (x, y) will be a multiple point of the curve and in case f_{x^2}, f_{xy}, f_{y^2} are not all zero, the points will be a node, cusp or a conjugate point according as

$$f_{xy}^2 - f_{x^2}f_{y^2} > 0, = 0, \text{ or, } < 0.$$

In case f_{x^2}, f_{xy}, f_{y^2} are all zero, the point (x, y) will be a multiple point of order higher than two.

Ex. 1. Find the nature of the singular points of the curve $ax^2+by^2-cxy=0$. **(C. H. 1961)**

Let $\quad f(x, y) = ax^2 + by^2 - cxy = 0$

Then $\quad f_x = 2ax - cy,\ f_y = 2by - cx$

Hence, (0, 0) satisfy all the equations

$$ax^2 + by^2 - cxy = 0$$

$$2ax - cy = 0$$

and $$2by - cx = 0$$

$\therefore$ Origin is a double point of the given curve.

Now origin is a node, cusp or a conjugate point according as

$$f_{xy}^2(0, 0) - f_{xx}(0, 0)\, f_{yy}(0, 0) > 0, = 0 \text{ or, } < 0.$$

i.e., according as

$$(-c)^2 - (2a)(2b) > 0, = 0 \text{ or, } < 0$$

i.e., according as

$$c^2 - 4ab > 0, = 0 \text{ or, } < 0.$$

Thus, origin is a node, if $c^2 > 4ab$

a cusp, if $c^2 = 4ab$

and a conjugate point if $c^2 < 4ab$.

Ex. 2. Find the nature of the singular points of the curve $(y-2x)^2 - x^5 = 0$. **(C. H. 1961)**

Let $\quad f(x, y) = (y-2x)^2 - x^5 = 0$

$$= y^2 - 4xy + 4x^2 - x^5 = 0$$

$\therefore \quad f_x = -4y + 8x - 5x^4$

$\quad\quad f_y = 2y - 4x$

Hence, (0, 0) satisfy all the three equations

$$f(x, y)=0,\ f_x=0,\ f_y=0$$

$\therefore$ Origin is a double point.

Again $f_{xx}=8-20x^3$, $f_{yy}=2$, $f_{xy}=-4$

$\therefore$ At the origin

$$f_{xx}(0, 0)=8,\ f_{yy}(0, 0)=2,\ f_{xy}(0, 0=-4.$$

Since, here $\{f_{xy}(0, 0)\}^2=f_{xx}(0, 0).f_{yy}(0\ 0)$.

The origin is a cusp.

11.10. To determine the tangents at the points where the curve attains its singularities.

If the equation of the curve does not contain the constant and the first degree term, then it is evidently satisfied by (0, 0) and so in that case origin is a multiple point of the curve. The equation of the tangent or tangents at the origin is usually obtained by equating to zero the terms of the lowest degree in the equation of the curve.

If (α, β) be a multiple point other than the origin, then we can conveniently, find the gradient of the curve at this point from the equation

$$f_{yy}\left(\frac{dy}{dx}\right)^2+2f_{xy}\left(\frac{dy}{dx}\right)+f_{xx}=0.$$

and consequently the equation of the tangents at (α, β) can be obtained by using the formula

$$y- \ =\frac{dy}{dx}\left(x-\alpha\right).$$

There is, however, a third method to find the tangent at (α, β). In this method, we shift the origin at (α, β) through parallel displacement by $x=X+\alpha$, $y=Y+\beta$ and the equation of the tangent at the new origin is obtained by equating the lowest degree terms of the transformed equation.

Ex. 1. Examine the singularities of the curve

$$\mathbf{x^4-4ax^3-2ay^3+4a^2x^2+3a^2y^2-a^4=0}$$

and find the equation of the tangents at singular points.

(C. H. 1962)

Let $f(x, y)=x^4-4ax^3-2ay^3+4a^2x^2+3a^2y^2-a^4=0$.

Since, $f(x, y)=0$ has a constant term, origin cannot be a singular point. To find the singularities, other than the origin, we write

$$f_x=4x^3-12ax^2+8a^2x\ ;\ f_y=-6ay^2+6a^2y.$$

For multiple point we must have $f_x=0$, $f_y=0$.

$\therefore$ From 1st equation

$$4x(x^2-3ax+2a^2)=0 \quad \text{or,} \quad 4x(x-a)(x-2a)=0$$

$$\Rightarrow \quad x=0,\ a,\ 2a$$

From the 2nd equation

$$6ay(a-y)=0$$

$$\Rightarrow \quad y=0,\ a$$

Now, we observe that the points $(0, a)$ and $(a, 0)$ simultaneously satisfy the equations

$$f(x, y)=0,\ f_x=0,\ f_y=0.$$

Hence, $(0, a)$ and $(a, 0)$ are two multiple points of the curve.

To find the nature of the singularities at $(0, a)$, we write,

$$\therefore \qquad f_{xx}=12x^2-24ax+8a^2$$

$$f_{xx}(0, a)=8a^2$$

$$f_{yy}=-12ay+6a^2$$

$$\therefore \quad f_{yy}(0, a)=-6a^2.$$

Also $f_{xy}=0 \qquad \therefore\ f_{xy}(0, a)=0.$

Now, since at $(0, a)$

$$f_{xy}^2-f_{xx}\cdot f_{yy}=48a^4>0,$$

the point $(0, a)$ is a node of the curve.

To find the tangents at $(0, a)$ we write

$$f_{yy}\left(\frac{dy}{dx}\right)^2+2f_{xy}\left(\frac{dy}{dx}\right)+f_{xx}=0$$

or, $$-6a^2\left(\frac{dy}{dx}\right)^2+8a^2=0$$

$$\Rightarrow \quad \frac{dy}{dx}=\pm\frac{2}{3}\sqrt{3}.$$

$\therefore$ Equation to the tangents at $(0, a)$ are

$$y-a=\pm\frac{2\sqrt{3}}{3}x.$$

Alternative method to find the tangent at (0, a)

Here, we shift the origin at $(0, a)$ by

$$x=X+0, \qquad y=Y+a.$$

So, the transformed equation becomes

$$X^4-4aX^3+4a^2X^2+3a^2(Y+a)^2-2a(Y+a)^3-a^4=0.$$

or, $X^4-4aX^3+4a^2X^2-3a^2Y^2-2aY^3=0$

Equating lower degree terms

$$4a^2X^2-3a^2Y^2=0$$

or, $3Y^2=4X^2$

or, $Y=\pm\frac{2}{\sqrt{3}}X$ which gives the tangents at the new origin.

$\therefore$ Tangent at $(0, a)$ are

$$y-a=\pm\frac{2}{\sqrt{3}}x=\pm\frac{2\sqrt{3}}{3}x.$$

Again, to find the nature of the singularities at $(a, 0)$, we observe that

$$f_{xx}(a, 0)=-4a^2\ ;\ f_{yy}(a, 0)=6a^2$$

and $\quad f_{xy}(a, 0)=0$

$\therefore$ At $(a, 0)$, $f_{xy}^2-f_{xx}.f_{yy}=24a^4>0.$

$\therefore$ The point $(a, 0)$ is also a node.

To find the tangent at $(a, 0)$

$$6a^2\left(\frac{dy}{da}\right)^2=4a^2$$

$$\Rightarrow \quad \frac{dy}{dx}=\pm\tfrac{1}{3}\sqrt{6}.$$

$\therefore$ Tangents at $(a, 0)$ are

$$y=\pm\tfrac{1}{3}\sqrt{6}(x-a).$$

11.11. Species of Cusps.

A point through which there pass two branches of the curve with a common tangent at that point is called a cusp. It will be seen that there may be five different ways in which the two branches of the curve lie in relation to the common tangent.

A cusp is called a single or double cusp according as the two branches lie on the same side or on different sides of the common normal at the cusp. Again, the cusp is called the first species or of the second species according as the two branches lie on the different or the same side of the common tangent at the cusp.

(1) Consider $y^2=x^3$ which has a cusp at (0, 0) with a common tangent $y=0$.

Here, for any negative value of x, y is imaginary. So, no part of the curve lies on the left side of the normal at (0, 0). Hence, the cusp is single.

Again, since, $y=\pm x^{3/2}$, for any positives values of x, there are two values of y equal and opposite. So, the two branches of the curve lie on opposite side of the common normal $y=0$. Hence, the cusp is of first species.

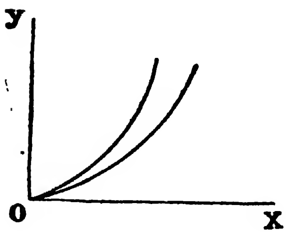
Hence, for the curve $y^2=x^3$, origin (0, 0) is a single cusp of the first species.

(2) Consider $(y-2x^2)^2=x^3$ i.e., $y=2x^2\pm x^{3/2}$ where the two signs correspond to the two branches of the curve.

Here, (0, 0) is a cusp with $y=0$ as the cuspidal tangent.

For any negative value of x, y is imaginary. So, no part of the curve lies on the negative sides of the x axis and evidently the two branches of the curve lie on the same side of the common normal $x=0$. Therefore, the cusp is single.

Again, $2x^2 > x^{5/2}$ if $2 > x^{\frac{1}{2}}$ i.e., if $4 > x$ i.e., if $x < 4$.

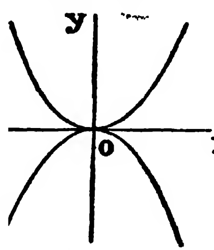


Thus, for any value of x in $0<x<4$, the values of y for both the curves $y=2x^2+x^{5/2}$ and $y=2x^2-x^{5/2}$ are positive.

$\because$ for values of x lying between 0 and 4, both the branches lie above the x axis and so the cusp is of the second species.

Hence, for the curve $y=2x^2 \pm x^{5/2}$, the origin (0, 0) is a single cusp of the second species.

(3) Consider $y^2-x^4=0$ which has a cusp at (0, 0) with a common tangent $y=0$.



Here, two branches $x^2=y$, $x^2=-y$ are two parabolas each one extending to both sides of the common normal $x=0$. Therefore, origin is a double cusp.

Again, the two parabolas lie on different sides of the cuspidal tangent $y=0$ and so the cusp is of the first species.

Hence, for the curve $y^2-x^4=0$, the origin (0, 0) is a double cusp of the first species.

(4) Consider $x^6-byx^4-b^3x^2y+b^4y^2=0$ which has a cusp at (0, 0) with the cuspidal tangent $y=0$.

The given equation is $b^4y^2-(bx^4+b^3x^2)y+x^6=0$

$$\therefore \quad y=\frac{(bx^4+b^3x^2) \pm \sqrt{(bx^4+b^3x^2)^2-4b^4x^6}}{2b^4} \quad \ldots \quad (1)$$

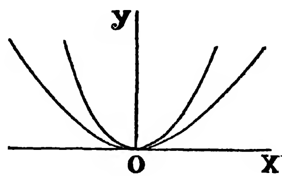
The leading term of the expansion under the radical sign is b^2x^8 which is positive whether x is positive or negative.

So, for x near the origin

$(bx^4+b^3x^2)^2-4b^4x^6$ has the same sign as b^2x^8 which is always positive.

$\therefore$ $(bx^4+b^3x^2)^2-4b^4x^6$ is positive for x near the origin

$\Rightarrow$ y is real for values of x near the origin.



Again, for positive or negative values of x, near the origin

$$(bx^4+b^3x^2)^2-4b^4x^6<(bx^4+b^3x^2)^2$$

$$\Rightarrow \quad \sqrt{(bx^4+b^3x^2)^2-4b^4x^6}<(bx^4+b^3x^2).$$

So, from (1), it follows that for values of x near the origin both the values of y are positive.

Hence, the origin (0, 0) is a double cusp of the second species.

Ex. 1. Examine the position and nature of the cusp of the semi cubical parabola

$$27ay^2=4(x-2a)^3. \qquad \textbf{(C. H. 1965 old)}$$

Let $f(x, y)=27ay^2-4(x-2a)^3$.

Then $f_x=-12(x-2a)^2$; $f_y=54ay$.

$f_{xx}=-24(x-2a)$; $f_{yy}=54a$ and $f_{xy}=0$.

For multiple points, we must have

$$f(x, y)=0, \quad f_x=0, \quad f_y=0$$

Now $f_y=0$, $f_x=0$ give

$$x=2a, \; y=0$$

Also $(2a, 0)$ satisfy $f(x, y)=0$

Hence, $(2a, 0)$ is the multiple point.

Again $f_{xx}(2a, 0)=0$, $f_{yy}(2a, 0)=54a$ and $f_{xy}(2a, 0)=0$

This gives

$$f_{xy}^2-f_{xx}\,f_{yy}=0$$

$\Rightarrow$ **The multiple point is a cusp.**

Now transfering the origin at $(2a, 0)$ by the equation $x=X+2a$, $y=Y$, the transferred equation becomes $27aY^2 =4X^3$ with respect to the new origin this equation has a cusp at $(0, 0)$ with a cuspidal tangent $X=0$.

Since, for any negative value of X, Y is imaginary, no part of the curve lies on the left side of the normal $X=0$. Hence, the cusp is single.

Again,

$$\because \quad Y=\pm\frac{2}{3\sqrt{3a}}X^{3/2}$$

for each positive value of X, there are two values of Y which are equal and opposite. So, the two branches of the curve lie on different sides of the common tangent $Y=0$. Hence, the cusp is of the first species.

Hence, for the given curve $(2a, 0)$ is a single cusp of the first species.

Ex. 2. Determine the position of the double points on the curve

$$\mathbf{x^2(x-2)^3-y(y-6)-9=0.} \qquad \textbf{(C. H. 1966)}$$

$$f(x, y)=x^2(x-2)^3-y(y-6)-9=0$$

$$\therefore \quad f_x=2x(x-2)^3+3x^2(x-2)^2$$
$$=x(x-2)^2(5x-4)$$
$$f_y=-(y-6)-y=-2y+6$$

Also $f_{xx}=(x-2)^2(5x-4)+2x(x-2)(5x-4)+5x(x-2)^2$

$$=(x-2)^2(10x-4)+2x(x-2)(5x-4)$$

$$f_{yy}=-2$$

and $f_{xy}=0$.

Now, for double points $f_x=0$, $f_y=0$

This gives $x=0, 2, \frac{4}{5}$ and $y=3$

$\therefore$ The points are $(0, 3)$, $(2, 3)$, $(\frac{4}{5}, 3)$ of which $(\frac{4}{5}, 3)$ does not satisfy $f(x, y)=0$.

$\therefore$ The multiple points are

$$(0, 3) \text{ and } (2, 3)$$

Now $f_{xx}(0, 3)=16, \quad f_{yy}(0, 3)=-2$

and $f_{xy}(0, 3)=0$

$\therefore \quad f^2{}_{xy}-f_{xx}\, f_{yy}=-32<0$

$\Rightarrow(0, 3)$ is a conjugate point.

Again, $f_{xx}(2, 3)=0, \quad f_{yy}(2, 3)=-2$ and $f_{xy}(2, 3)=0$

$\therefore \quad f^2{}_{xy}-f_{xx}\cdot f_{yy}=0$

$\Rightarrow(2, 3)$ is a cusp.

Transferring the origin to the point (2, 3) by parallel displacement by the equation $x=X+2$, $y=Y+3$ the transformed equation becomes

$$(X+2)^2X^3-(Y+3)(Y-3)-9=0$$

or, $X^3(X+2)^2-Y^2=0 \qquad \cdots \qquad \cdots \qquad (1)$

$\Rightarrow(0, 0)$ is a cusp of (1) with its cupsidal tangent $Y=0$.

Equation to the normal at $(0, 0)$ is $X=0$. If X be negative, Y becomes imaginary, so, no part of the curve lies on the left side of the normal $X=0$

$\Rightarrow(0, 0)$ is a single cusp.

Again, $Y=\pm X^{\frac{3}{2}}(X+2)$

$\therefore$ for any positive value of X, Y has two values which are equal and opposite. So, the two branches lie on the opposite sides of the common tangent $Y=0$

$\Rightarrow$ (0, 0) is a cusp of the first species.

Hence, (2, 3) is single cusp of the first species.

Ex. 8. Show that each of the curves

$$(x \cos \alpha-y \sin \alpha-b)^3=(x \sin \alpha+y \cos \alpha)^2$$

where α is variable, has a cusp, and all the cusps lie on a circle. **(C. H. 1962)**

Rotating the axes though an angle α by the equation $x'=x \cos \alpha-y \sin \alpha$, $y'=x \sin \alpha+y \cos \alpha$ the transformed equation becomes

$$(x'-b)^3=y'^2 \qquad \cdots \qquad \cdots \qquad (1)$$

Again, transferring the origin to $(b, 0)$ by parallel displacement, by the equation $x' = X + b$, $y' = Y$, the equation (1) again becomes

$$X^3 = Y^2 \quad \cdots \quad \cdots \quad (2)$$

$\Rightarrow (0, 0)$ is a cusp with respect to the new origin with the cupsidal tangent $Y = 0$.

The normal to the curve at $(0, 0)$ is $X = 0$. For any negative value of X, Y is imaginary,

$\therefore$ No part of the curve lies on the left side of the common normal $X = 0$.

$\Rightarrow (0, 0)$ is a single cusp of (2).

Again $\quad Y = \pm X^{\frac{3}{2}}$

$\therefore$ for any positive value of X, Y has two values which are equal and opposite.

$\therefore$ the two branches of the curve lie on opposite side of the tangent $Y = 0$

$\Rightarrow (0, 0)$ is a cusp of the first species of (2)

Thus, the point given by

$$x \cos \alpha - y \sin \alpha = b, \; x \sin \alpha + y \cos \alpha = 0 \quad \cdots \quad \cdots \quad (3)$$

is a single cusp of the first species.

Eliminating α from the two equations in (3), we get,

$$x^2 + y^2 = b^2 \text{ which is a circle.}$$

Hence, all the cusps lie on the circle $x^2 + y^2 = b^2$.

Exercise 11

1. Show that $(0, 1)$ and $(8, e^2)$ are the points of inflexion of the curve

$$x = (\log y)^3.$$

2. Show that origin is the point of inflexion of the curve

(*i*) $a^2y^2 = a^2x^2 - x^4$ and (*ii*) $y = x^3 \log(1 - x)$.

3. Prove that the curve

$$cy^2 = x(a - c)(x - d)$$

has two and only two points of inflexion.

4. Show that the points of inflexion on the curve $y^2 = (x - a)^2(x - b)$ lie on the line $3x = 4b - a$.

5. Show that origin is a node, a cusp or a conjugate point on the curve

$$y^2=ax^2(1+x)$$

according as a is positive, zero or negative.

6. Show that origin on the curve

$$x^3=y^2(a-x)$$

is a single cusp of the first species.

7. Prove that the curve

$$x^3+y^3-3xy=0$$

has a double point at the origin and find the tangent there.

8. Show that the point $(c, 0)$ on the curve

$$(x-c)^3-y^2=0$$

is a cusp of the first species. (C. H. 1960)

9. Show that the curve

$$\{x^2-(y-c)^2\}(x-2)+x=0$$

has a node at $(1, c)$ and an isolated point at $(\frac{1}{3}, c)$, (C. H. 1960)

10. Show that the curve

$$y^3=(x-a)^2(2x-a)$$

has a single cusp at $(a, 0)$.

11. Show that (0. 2) on the curve

$$y=2+x(1+x+x^{\frac{3}{2}})$$

is a single cusp of second species.

12. Find the double point on the curve

$$x^6-2yx^4-8x^3y+16y^2=0$$

and show that it is a double cusp of second species.

13. Prove that the curve

$$x^3+y^3=cx^2$$

has a point of inflexion at $x=c$ and a cusp of the first species at the origin.

14. Show that origin on the evolute of the parabola $y^2=4ax$ is a single cusp of the first species.

15. Examine the nature of the origin on the curve

$$(2x+y)^2-6xy(2x+y)-7x^3=0$$

and show that it is a cusp. (C. H. 1964 old)

16. Find the nature of the double point of the curve

$$y^2=(x-2)^2(x-5)$$

and show that the curve has two real points of inflexion and that they subtend a right angle at the double point.

17. Define a point of inflexion on a curve and show that origin is a point of inflexion of the conic

$$y=axy+by^2+cx^3.$$

(B. H. 1967)

CHAPTER XII

ENVELOPES

12.1. Family of curves and its parameter.

If the equation to a straight line or a curve be a function of three variabls x, y and α then, the equation $f(x, y, \alpha)=0$ determines a straight line or a curve corresponding to each particular value assigned to α.

For different values of α, the equation will give different straight lines or curves. The totality of these straight lines or curves is called family of straight lines or curves, and the variable α is called the parameter of the family.

12.2. Definition of Envelope.

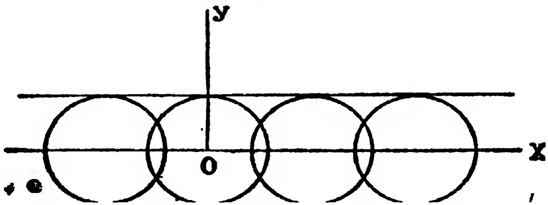
If each member of the family of straight lines or curves $C \equiv F(x, y, \alpha)=0$ touches a fixed straight line or a curve E, then E is called the envelope of the family of straight lines or curves C. Conversely, E is the envelope of C if each point of E is touched by some member of the family C.

Illustrations

(1) The equation to a straight line in the normal form is $x \cos \alpha + y \sin \alpha = a$ where a is the length of the normal from the origin on the line and α is the angle between the normal and the positive direction of x axis. If a is kept constant and α is allowed to vary, then we shall get a family of straight lines each one of which touching a fixed circle $x^2+y^2=a^2$ at $(a \cos \alpha, a \sin \alpha)$. Hence, in this case the family $F(x, y, \alpha)=0$ is a family of straight lines and its envelope is a curve i.e., a circle $x^2+y^2=a^2$.

(2) The family of circles $(x-\alpha)^2+y^2=a^2$ where a is fixed and α is a variable parameter represent a series of

equal circles each of radius a and having their centres at different points on the x axis. These circles are touched



by the lines $y = \pm a$. Hence, here envelope of the family of curves which are circles, are straight lines $y = \pm a$.

12.3. Envelope of Straight Lines.

The α eliminant curve from $F(x, y, \alpha) = 0$ and $\frac{dF}{d\alpha}(x, y, \alpha) = 0$ is the envelope of a family of straight lines $F(x, y, \alpha) = 0$ with α as its parameter.

Let $F(x, y, \alpha) \equiv y - f(\alpha)x - \phi(\alpha) = 0$ where α is a parameter, be a family of straight line.

$$F(x, y, \alpha) = 0$$

$$\Rightarrow y = f(\alpha)x + \phi(\alpha) \quad \cdots \quad \cdots \quad (1)$$

$$\frac{dF}{d\alpha}(x, y, \alpha) = 0$$

$$\Rightarrow 0 = f'(\alpha)x + \phi'(\alpha) \quad \cdots \quad \cdots \quad (2)$$

From (2) $x = -\dfrac{\phi'(\alpha)}{f'(\alpha)} = \xi(\alpha)$ (say) $\quad \cdots \quad \cdots \quad (3)$

Putting this value of x in (1)

$$y = -f(\alpha)\frac{\phi'(\alpha)}{f'(\alpha)} + \phi(\alpha) = \zeta(\alpha) \text{ say} \quad \cdots \quad \cdots \quad (4)$$

Thus, α eliminant curve of (1) and (2) is the same as the curve where parametric equations are given by

$$\left.\begin{aligned} x &= \xi(\alpha) \\ y &= \zeta(\alpha) \end{aligned}\right\} \quad \cdots \quad \cdots \quad (5)$$

Now, equation to a tangent at α to the curve (5) is

$$y - \zeta(\alpha) = \frac{\zeta'(\alpha)}{\xi'(\alpha)}\left\{x - \xi(\alpha)\right\}$$

$$= f(\alpha)\{x - \xi(\alpha)\} \quad \text{from (5)}$$

or, $y+f(\alpha)\dfrac{\phi'(\alpha)}{f'(\alpha)}-\phi(\alpha)=f(\alpha)\left\{x+\dfrac{\phi'(\alpha)}{f'(\alpha)}\right\}$

or, $y-f(\alpha)x-\phi(\alpha)=f(\alpha)x-f(\alpha)x.$

or, $y=f(\alpha)x+\phi(\alpha)$, which is the same as the arm equation of the given family of straight lines. Thus, the curve where paramatric equation is given by (5) is touched by the family of straight lines $y=f(\alpha)x+\phi(\alpha)$.

So, it follows that the curve (5) which is the α eliminant of $F=0$ and $\dfrac{dF}{d\alpha}=0$, is the envelope of the family of straight lines $y=f(\alpha)x+\phi(\alpha)$, i.e., $F\ x, y, \alpha)=0$

12.4. Envelope of curve lines.

The envelope of the family of curves $f(x, y, \alpha)=0$ with α as the variable parameter is the α eliminant curve from

$$f(x, y, \alpha)=0 \text{ and } f_\alpha(x, y, \alpha)=0$$

where f_α is the partial derivative of f with respect to α.

Suppose, α eliminant curve is the envelope of

$$f(x, y, \alpha)=0 \quad \cdots \quad \cdots \quad (1)$$

But α eliminant is obtaind by solving for x, y from

$$f(x, y, \alpha)=0 \text{ and } f_\alpha(x, y, \alpha)=0 \quad \cdots \quad (2)$$

$\therefore$ Equation of envelope must be of the parametric form

$$\left.\begin{array}{l} x=\phi(\alpha) \\ y=\psi(\alpha) \end{array}\right\} \quad \cdots \quad \cdots \quad (3)$$

The equation to the tangent at any point (x, y) in (1) is

$$(X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}=0 \quad \cdots \quad \cdots \quad (4)$$

Also, equation to the tangent at the point (x, y) on (3) is

$$Y-y=\frac{dy}{dx}(X-x)$$

$$=\frac{\dfrac{dy}{d\alpha}}{\dfrac{dx}{d\alpha}}(X-x)$$

$$\text{or,}\quad (X-x)\frac{dy}{d\alpha}-(Y-y)\frac{dx}{d\alpha}=0 \quad \cdots \quad (5)$$

Since, (4) and (5) are coincident straight lines the co-efficient of X and Y must be proportional.

$$\therefore \quad \frac{\dfrac{\partial f}{\partial x}}{\dfrac{dy}{d\alpha}}=\frac{\dfrac{\partial f}{\partial y}}{-\dfrac{dx}{d\alpha}}$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}\frac{dx}{d\alpha}+\frac{\partial f}{\partial y}\frac{dy}{d\alpha}=0 \quad \cdots \quad (6)$$

Again, differentiating $f(x, y, \alpha)=0$. w. r. to α and considering x, y as functions of α, we get,

$$\frac{\partial f}{\partial \alpha}+\frac{\partial f}{\partial x}\frac{dx}{d\alpha}+\frac{\partial f}{\partial y}\frac{dy}{d\alpha}=0$$

$$\Rightarrow \quad \frac{\partial f}{\partial \alpha}=0 \qquad \text{by (6).}$$

Thus, if (3) be the envelope of (1), then the two equations $f(x, y, \alpha)=0$ and $f_\alpha(x, y, \alpha)=0$ must hold simultaneously. Hence, the result.

Cor. 1. The envelope of the family of curves $A\alpha^2+B\alpha+C=0$ where α is the variable parameter and A, B, C are linear functions of x, y is *the curve* $B^2=4AC$.

For differentiating $A\alpha^2+B\alpha+C=0 \quad \cdots \quad (1)$
with respect to α

$$2A\alpha+B=0 \qquad \text{i.e.,} \quad \alpha=-B/2A$$

Putting this value of α in (1), the required envelope is

$$A\frac{B^2}{4A^2}-\frac{B^2}{2A}+C=0$$

$$\text{or,} \quad B^2-2B^2+4AC=0$$

$$\text{or,} \quad B^2=4AC.$$

Cor. 2. *Every curve is the envelope of its tangent.*

We know that $y=mx+\frac{a}{m}$ are tangents to the parabola

$y^2=4ax$ for all values of m. So $y^2=4ax$ is the envelope of the family of straight lines $y=mx+\frac{a}{m}$.

To prove this let us write the family of straight lines $y=mx+\frac{a}{m}$ in the quadratic form

$$m^2x-my+a=0.$$

Then, the envelope is

$$(-y)^2=4(x)(a)$$

i.e., $$y^2=4ax.$$

Cor. 3. Since normals to a curve are the tangents to its evolute, it follows that *the evolute of a curve is the envelope of its normals.*

We know that at a point 'm' on the parabola $y^2=4ax$, the equation to the normal is $y=mx-2am-am^3$ $\cdot\cdot$ (1)

To find the envelope of (1), we differentiate (1) w.r. to m so that

$$0=x-2a-3am^2 \quad \text{or,} \quad m^2=\frac{x-2a}{3a}$$

Now from (1) $y=m\{(x-2a)-am^2\}$

$$=m\left\{(x-2a)-\frac{x-2a}{3}\right\}=\frac{2}{3}m(x-2a)$$

$$\therefore \quad y^2=\frac{4}{9}m^2(x-2a)^2$$

$$=\frac{4}{9}\cdot\frac{(x-2a)^3}{3a}$$

or, $27ay^2=4(x-2a)^3$ which is the envelope of the normals.

Hence, the evolute of the parabola $y^2=4ax$ is

$$27ay^2=4(x-2a)^3.$$

Cor. 4. If $\phi(x, y, \alpha)=0$ be the equation to the normal to a family of curves $f(x, y, \alpha)=0$ where α is a variable parameter, then α eliminant curve from $\phi(x, y, \alpha)=0$ and

$\frac{\partial\phi}{\partial\alpha}=0$ will give the evolute of the family. But evolute is the locus of the centres of curvature. Hence, *the centre of curvature may be obtained by solving for* x, y *the equations* $\phi(x, y, \alpha)=0$ and $\frac{\partial\phi}{\partial\alpha}$ *in terms of* α.

The normal to the curve $x^{2/3}+y^{2/3}=a^{2/3}$ at θ is

$$x \cos \theta-y \sin \theta=a \cos 2\theta. \quad \cdots \quad (1)$$

Differentiating (1) w. r. to θ

$$x \sin \theta+y \cos \theta=2a \sin 2\theta \quad \cdots \quad (2)$$

Solving (1) and (2)

$$x=a(\cos \theta+\sin \theta \sin 2\theta), \quad y=a(\sin 2\theta \cos \theta+\sin \theta)$$

But $x=a \cos^3\theta$ and $y=a \sin^3\theta$

$\therefore$ If $(\bar{x}, \bar{y})$ be the co-ordinates of the centre of curvature, then

$$\begin{aligned}\bar{x}&=a \cos \theta(1+2 \sin^2\theta)\\ &=a\frac{x^{1/3}}{a^{1/3}}\left(1+2\frac{y^{2/3}}{a^{2/3}}\right)\\ &=a^{2/3}\left(x^{1/3}+2\frac{x^{1/3}\ y^{2/3}}{a^{2/3}}\right)\\ &=a^{2/3}\ x^{1/3}+2x^{1/3}\ y^{2/3}\\ &=(x^{2/3}+y^{2/3})x^{1/3}+2x^{1/3}\ y^{1/3}\\ &=x+3x^{1/3}\ y^{2/3}.\end{aligned}$$

Similarly, $\bar{y}=y+3x^{2/3}y^{1/3}$.

12.5. Singular points on a Curve.

A point (a, b) on the curve $f(x, y, \alpha)=0$ where α is fixed, is said to be a singular point of the curve if it satisfies simultaneously the three equations

$$f(x, y, \alpha)=0, \quad f_x=0, \quad f_y=0.$$

Again, the point (a, b) is called an ordinary point if it does not satisfy at least one of the two equations $f_x=0$ $f_y=0$.

Theorem I

Any singular point of any curve of a given family is a point on its envelope.

Let $f(x, y, \alpha)=0$ (where α is fixed) $\cdots$ (1)
be any member of a family of curves.

Let (a, b) be a singular point of (1), then (a, b) will simultaneously satisfy the three equations.

$$f(x, y, \alpha)=0, \quad f_x=0, \quad f_y=0. \qquad \cdots (2)$$

Now from (1)

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial \alpha}d\alpha=0$$

$$\Rightarrow \quad \frac{\partial f}{\partial \alpha}d\alpha=0 \quad \text{using (2)}$$

$$\Rightarrow \quad \frac{\partial f}{\partial \alpha}=0$$

So, it follows that (a, b) satisfies the two equations

$$f(x, y, \alpha)=0$$

$$\text{and} \quad \frac{\partial f}{\partial \alpha}=0$$

Thus, the locus of the singular points of the curves of a family is a point of its envelope.

Ex. Show that the locus of the singular point on the curve $x^2(x-a)+(x+a)(y-m)^2=0$ where a is a constant and m is a variable parameter, is a part of its envelope.

$$f(x, y, m)=x^2(x-a)+(x+a)(y-m)^2=0$$
$$f_x=3x^2-2ax+(y-m)^2$$
$$f_y=2(y-m)$$

$\because$ $(0, m)$ satisfy the three equations

$$f(x, y, m)=0, \quad f_x=0, \quad f_y=0$$

$(0, m)$, where m is variable, is the singular point of the curve.

$\Rightarrow$ Locus of the singular point on the curve is the y axis.

To find the envelope of the given curve, we differentiate it w. r. to m, so that

$$-2(x+a)(y-m)=0$$

i.e., $y-m=0$

$\therefore$ Eliminating m, the envelope is

$$x^2(x-a)=0$$

which gives two lines $x=0$ and $x=a$

But $x=0$ is the y axis i e., locus of the singular point and $x=a$ is outside the locus.

Thus, we find that the locus of the singular point is a part of the envelope.

12.6. Characteristic points of a curve

A point (a, b) of a family of curve $f(x, y, \alpha)=0$ if does not satisfy at least one of the two equations $f_x=0$, $f_y=0$ (i.e., if the point be an ordinary point), then it is called a characteristic point of the family provided it satisfies simultaneously the two equations

$$f(x, y, \alpha)=0 \text{ and } \frac{\partial f}{\partial \alpha}(x, y, \alpha)=0 \qquad \cdots \quad (1)$$

Now (a, b) will satisfy both the equation (1), provided the two equations

$$f(x, y, \alpha)=0 \text{ and } \frac{\partial f}{\partial x}(x, y, \alpha)=0 \text{ intersect.}$$

Again, since (a, b) is an ordinary point, it can not be a singular point of the family.

Hence, the characteristic points are those points of intersection of $f=0$ and $f_\alpha=0$ which are not singular points of the family $f(x, y, \alpha)=0$.

The locus of the characteristic points of the family of curves is sometimes called the envelope of the family.

In practice, it is not possible to solve the two equations $f=0$ and $f_\alpha=0$ unless one of the three variables is kept constant. In finding characteristics points, we are to find x, y and so α must be kept fixed.

Theorem I

The envelope of the family of curves $f(x, y, \alpha)=0$ touches every member of the family at its characteristic point.

(C. H. 1971)

Let $P(x, y)$ be a characteristic point of the family $f(x, y, \alpha)=0$ where α is fixed$=\alpha_1$ (say).

Then, the slope of the tangent at (x, y) to the curve when $\alpha=\alpha_1$ is obtained from

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0$$

Thus, if in the differentiation α is kept constant at α_1, then the slope of the tangent at (x, y) is

$$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} \quad \ldots \quad \ldots \quad (1)$$

The envelope of the family is obtained by eliminating α from

$$f(x, y, \alpha)=0$$
$$\text{and } f'_\alpha(x, y, \alpha)=0$$

Let $\left.\begin{array}{l} x=\phi(\alpha) \\ y=\psi(\alpha) \end{array}\right\} \quad \ldots \quad \ldots \quad (2)$

be the parametric equation of the envelope obtained by solving the above two equations for x, y in terms of α.

So, the equation of the envelope can be put in the form

$$f\{\phi(\alpha), \psi(\alpha), \alpha\}=0$$

This gives, on total differentiation,

$$\frac{\partial f}{\partial x}\phi'(\alpha)+\frac{\partial f}{\partial y}\psi'(\alpha)+\frac{\partial f}{\partial \alpha}=0$$

$$\Rightarrow \frac{\partial f}{\partial x}\phi'(\alpha)+\frac{\partial f}{\partial y}\psi'(\alpha)=0 \quad \because \quad f_\alpha(x, y, \alpha)=0$$

$$\Rightarrow \frac{\psi'(\alpha)}{\phi'(\alpha)}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} \quad \ldots \quad (3)$$

But, the slope of the tangent to the envelope (2) at $\alpha=\alpha_1$ is $\dfrac{dy}{dx}=\dfrac{\psi'(\alpha)}{\phi'(\alpha)} \quad \ldots \quad \ldots \quad (4)$

Hence, from (1), (3) and (4), we find that at a characteristic point $P(x, y)$ the slope of the tangent to the family and the envelope are the same.

$\Rightarrow$ two tangents coincide.

Hence, the theorem.

Theorem 2

If it is possible to find a curve E such that it is touched by the members of the family[1] $C \equiv f(x, y, \alpha) = 0$, then E must be the envelope of C. (Converse of Theorem 1).

Let $P(x, y)$ be any point on the curve E. Then this point is also a point on some member of the family C.

$\therefore$ x, y must be functions of α and so we write

$$x = \phi(\alpha)$$
$$y = \psi(\alpha).$$

Since, these coordinates also satisfy $f(x, y, \alpha) = 0$, we can write

$$f\{\phi(\alpha), \psi(\alpha), \alpha\} = 0.$$

This gives on differentiation w. r. to α

$$\frac{\partial f}{\partial x}\phi'(\alpha) + \frac{\partial f}{\partial y}\psi'(\alpha) + \frac{\partial f}{\partial \alpha} = 0 \qquad \cdots \quad (1)$$

Now, the slope of the tangent at P to the curve C is

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}$$

Also, the slope of the tangent at P to the curve is

$$\frac{dy}{dx} = \frac{\psi'(\alpha)}{\phi'(\alpha)}$$

Since, the two curves touch at P, these slopes must be be equal.

$$\therefore \quad -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} = \frac{\psi'(\alpha)}{\phi'(\alpha)}$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}\phi'(\alpha) + \frac{\partial f}{\partial y}\psi'(\alpha) = 0.$$

So, from (1) we get

$$\frac{\partial f}{\partial x} = 0$$

$\Rightarrow$ E is the locus of the characteristic points. Hence, E is the envelope of C.

Ex. 1. Given that $x^{2/3}+y^{2/3}=c^{2/3}$ is the envelope of $x/a+y/b=1$. Find the necessary relation between a and b.
(C. H. 1963, 72)

Let $P\,(c\cos^3\phi, c\sin^3\phi)$ be any point on the envelope $x^{2/3}+y^{2/3}=c^{2/3}$.

Since, every point of the envelope is touched by some members of the family $\frac{x}{a}+\frac{y}{b}=1$, $P(c\cos^3\phi, c\sin^3\phi)$ is also a point on the family.

Now, tangent at P on the envelope is

$$y-c\sin^3\phi=\frac{3c\sin^2\phi\cos\phi}{3c\cos^2\phi(-\sin\phi)}(x-c\cos^3\phi).$$

$$=-\frac{\sin\phi}{\cos\phi}(x-c\cos^3\phi).$$

or, $y\cos\phi-c\sin^3\phi\cos\phi=-x\sin\phi+c\sin\phi\cos^3\phi.$

or, $x\sin\phi+y\cos\phi=c\sin\phi\cos\phi.$

or, $\dfrac{x}{c\cos\phi}+\dfrac{y}{c\sin\phi}=1.$ $\quad\cdots\quad$ (1)

Since, $\frac{x}{a}+\frac{y}{b}=1$ touches the curve at P, it must be identical with (1)

$\therefore\ a=c\cos\phi,\ b=c\sin\phi.$

Hence, $a^2+b^2=c^2$ is the required relation.

12.7. Envelope of a family of curves containing two parameters.

Suppose, equation to a family of curves be

$$f(x, y, \alpha, \beta)=0 \quad\cdots\quad\cdots\quad (1)$$

where α, β are two parameters connected by an equation

$$\phi(\alpha, \beta)=0 \quad\cdots\quad\cdots\quad (2)$$

Case I. Suppose equation (2) is easily solvable for β in terms of α.

i.e., let $\beta=\psi(\alpha)$.

Then, putting in value of β in (1), we get

$$f\{x, y, \alpha, \psi(\alpha)\}=0 \quad\cdots\quad\cdots\quad (3)$$

which reduces to a family of curves with one parameter α only.

So, eliminating α between (3) and $f_\alpha\{x, y, \alpha, \psi(\alpha)\}$ we can easily find the envelope of the given curve.

Case II. Differentiating (1) and (2) w. r. to α, β and regarding x, y as constants, we get

$$\frac{\partial f}{\partial \alpha}d\alpha+\frac{\partial f}{\partial \beta}d\beta=0$$

and $$\frac{\partial \phi}{\partial \alpha}d\alpha+\frac{\partial \phi}{\partial \beta}d\beta=0.$$

Eliminating $d\alpha$ and $d\beta$ from these two equations we get

$$\frac{\partial f}{\partial \alpha}\Big/\frac{\partial \phi}{\partial \alpha}=\frac{\partial f}{\partial \beta}\Big/\frac{\partial \phi}{\partial \beta} \quad \cdots \quad \cdots \quad (3)$$

Now, eliminating α, β from (1), (2) and (3), the equation to the envelope can be easily obtained.

Ex. 1. Find the envelope of a straight line of length k whose extremities lie on the coordinate axes. (C. H. 1967)

Let a and b be the intercepts made by the line of length k with the axes so that equation to the family of lines is

$$\frac{x}{a}+\frac{y}{b}=1$$

where a and b are parameters connceted by the relation

$$a^2+b^2=k^2.$$

First method. The equation $a^2+b^2=k^2$ is solvable for b in terms of a i.e., $b=\sqrt{k^2-a^2}$.

Putting this value of b, the equation to the family becomes

$$\frac{x}{a}+\frac{y}{\sqrt{k^2-a^2}}=1 \quad \cdots \quad \cdots \quad (1)$$

Differentiating this w. r. to a

$$-\frac{x}{a^2}-\tfrac{1}{2}y(k^2-a^2)^{-3/2}(-2a)=0$$

or, $$\frac{ay}{(k^2-a^2)^{3/2}}=\frac{x}{a^2} \quad \cdots \quad \cdots \quad (2)$$

Now eliminating a between (1) and (2), the required envelope is obtained thus :

From (1) $\frac{y}{\sqrt{k^2-a^2}}=1-\frac{x}{a}=\frac{a-x}{a}$

$$\text{or, } \frac{y^3}{(k^2-a^2)^{3/2}}=\frac{(a-x)^3}{a^3} \quad \cdots \quad \cdots \quad (3)$$

Dividing (2) by (3)

$$\frac{a}{y^2}=\frac{ax}{(a-x)^3}$$

$$\text{or, } (a-x)^3=xy^2$$

$$\text{or, } a=x+x^{\frac{1}{3}}y^{\frac{2}{3}}.$$

Now, from (1) $\frac{y^2}{k^2-a^2}=\left(1-\frac{x}{a}\right)^2$

$$\text{or, } \frac{a^2y^2}{k^2-a^2}=(a-x)^2$$

$$=x^{\frac{2}{3}}y^{\frac{4}{3}}$$

$$\text{or, } \frac{\left(x+x^{\frac{1}{3}}y^{\frac{2}{3}}\right)^2y^2}{k^2-\left(x+x^{1/3}y^{2/3}\right)^2}=x^{\frac{2}{3}}y^{\frac{4}{3}}$$

$$\text{or, } \frac{x^{\frac{2}{3}}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2.y^{\frac{2}{3}}}{k^2-x^{\frac{2}{3}}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2}=x^{\frac{2}{3}}$$

$$\text{or, } \left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2.y^{\frac{2}{3}}=k^2-x^{\frac{2}{3}}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2$$

$$\text{or, } \left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^3=k^2$$

$$\text{or, } x^{\frac{2}{3}}+y^{\frac{2}{3}}=k^{\frac{2}{3}}.$$

Second method. The family is

$$\frac{x}{a}+\frac{y}{b}=1 \quad \cdots \quad \cdots \quad (1)$$

where $\quad a^2+b^2=k^2 \quad \cdots \quad \cdots \quad (2)$

Differentiating (1) and (2) w.r. to a

$$-\frac{x}{a^2}da-\frac{y}{b^2}db=0$$

and $\quad 2ada+2bdb=0$

Eliminating da and db from these two equations

$$\frac{x}{a^3}=\frac{y}{b^3}$$

or, $\quad \dfrac{\frac{x}{a}}{a^2}=\dfrac{\frac{y}{b}}{b^2}=\dfrac{\frac{x}{a}+\frac{y}{b}}{a^2+b^2}=\dfrac{1}{k^2}$

$\therefore \quad x=\dfrac{a^3}{k^2}$ and $y=\dfrac{b^3}{k^2}$

and so $\quad x^{2/3}+y^{2/3}=\dfrac{a^2}{k^{4/3}}+\dfrac{b^2}{k^{4/3}}=\dfrac{a^2+b^2}{k^{4/3}}=\dfrac{k^2}{k^{4/3}}$

$=k^{2/3}$ is the required envelope.

Ex. 2. Find the envelope of the straight lines

$$\frac{x}{a}+\frac{y}{b}=1 \quad \text{where } a^n+b^n=c^n \qquad \text{(C. H. 1964)}$$

Differentiating the two equations w. r. to a

$$-\frac{x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0$$

and $\quad na^{n-1}+nb^{n-1}\dfrac{db}{da}=0.$

$\therefore$ Eliminating $\dfrac{db}{da}$ we get

$$\frac{\frac{x}{a^2}}{na^{n-1}}=\frac{\frac{y}{b^2}}{nb^{n-1}}$$

or, $\quad \dfrac{x/a}{a^n}=\dfrac{y/b}{b^n}=\dfrac{x/a+y/b}{a^n+b^n}=\dfrac{1}{c^n}.$

$\therefore \quad a^{n+1}=xc^n$ and $b^{n+1}=yc^n$

or, $\quad (xc^n)^{\frac{1}{n+1}}=a$ and $(yc^n)^{\frac{1}{n+1}}=b$

$$\therefore \quad (xc^n)^{\frac{n}{n+1}}+(yc^n)^{\frac{n}{n+1}}=a^n+b^n=c^n$$

or, $$c^{\frac{n^2}{n+1}}\left\{x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right\}=c^n$$

or, $x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}=c^{\frac{n}{n+1}}$ is the required envelope.

Ex. 3. Find the envelope of a variable circle which passes through the origin and whose centre lies on the ellipse $x^2/a^2+y^2/b^2=1$ (C. H. 1961)

Let the equation to the variable circle be

$$x^2+y^2-2\alpha x-2\beta y=0 \quad \cdots \quad (1)$$

whose centre (α, β) lies on $x^2/a^2+y^2/b^2=1$

$$\therefore \quad \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1 \quad \cdots \quad \cdots \quad (2)$$

Differentiating (1) and (2) *w. r.* to α

$$-2x-2y\frac{d\beta}{d\alpha}=0$$

and $$\frac{2\alpha}{a^2}+\frac{2\beta}{b^2}\frac{d\beta}{d\alpha}=0$$

Eliminating $\frac{d\beta}{d\alpha}$ from these two equations, we get

$$\frac{x}{\alpha/a^2}=\frac{y}{\beta/b^2}=k \text{ (suppose).}$$

$$\therefore \quad \alpha=\frac{a^2x}{k}, \quad \beta=\frac{b^2y}{k}.$$

Putting in (2)

$$\frac{a^2x^2}{k^2}+\frac{b^2y^2}{k^2}=1$$

or, $k^2=a^2x^2+b^2y^2$

$$\therefore \quad \alpha=\frac{a^2x}{\sqrt{a^2x^2+b^2y^2}}, \quad \beta=\frac{b^2y}{\sqrt{a^2x^2+b^2y^2}}.$$

So that putting these values of α and β in (1) the required envelope is

$$x^2+y^2=2\left\{\frac{a^2x^2}{\sqrt{a^2x^2+b^2y^2}}+\frac{b^2y^2}{\sqrt{a^2x^2+b^2y^2}}\right\}$$

$$=2\frac{a^2x^2+b^2y^2}{\sqrt{a^2x^2+b^2y^2}}=2\sqrt{a^2x^2+b^2y^2}$$

Squaring $(x^2+y^2)^2=4(a^2x^2+b^2y^2)$.

Ex. 4. From any point P on a parabola, PM and PN are drawn perpendiculars to the axis and the tangent at the vertex, show that the envelope of MN is another parabola.

(C. H. 1971)

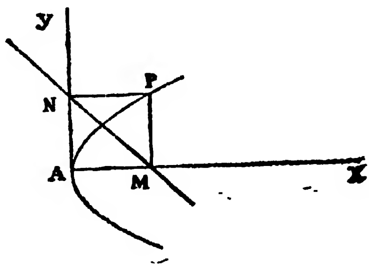
Let $P(\alpha, \beta)$ be any point on the parabola $y^2=4ax$

Then, $\beta^2=4a\alpha$ ... (1)

Coordinates of M and N are clearly $(\alpha, 0)$ and $(0, \beta)$

$\therefore$ Equation to MN is

$$\frac{x}{\alpha}+\frac{y}{\beta}=1. \quad \cdots \quad (2)$$



Differentiating (2) and (1) $w.$ $r.$ to α

$$-\frac{x}{\alpha^2}-\frac{y}{\beta^2}\frac{d\beta}{d\alpha}=0$$

and $\quad 2\beta\dfrac{d\beta}{d\alpha}=4a$

Eliminating $\dfrac{d\beta}{d\alpha}$ from these two equations

$$\frac{-\dfrac{x}{\alpha^2}}{4a}=\frac{y}{2\beta^3}.$$

or, $\quad -\dfrac{\dfrac{x}{\alpha}}{4a\alpha}=\dfrac{y}{2\beta^3}$

or, $\quad -\dfrac{\dfrac{x}{\alpha}}{\beta^2}=\dfrac{y}{2\beta^3} \qquad$ or, $\quad -\dfrac{x}{\alpha}=\dfrac{y}{2\beta}=k$ (suppose),

$$\therefore \quad \alpha=-\frac{x}{k}, \ \beta=\frac{y}{2k} \qquad \cdots \quad (3)$$

Putting in (1) $\dfrac{y^2}{4k^2}=-\dfrac{4ax}{k}$

$$\text{or,} \quad \frac{y^2}{k}=-16ax \qquad \text{or,} \quad k=-\frac{y^2}{16ax}$$

$\therefore$ From (3) $\alpha=\dfrac{16ax^2}{y^2}, \quad \beta=-\dfrac{8ax}{y}.$

Putting in (1), the required envelope is

$$\frac{y^2}{16ax}-\frac{y^2}{8ax}=1 \quad \text{or,} \quad y^2-2y^2=16ax$$

or, $y^2+16ax=0$ which is also a parabola.

Otherwise. Equation (2) can be written as

$$\frac{4ax}{\beta^2}+\frac{y}{\beta}=1 \qquad \left[\because \ \text{From (1)} \ \alpha=\frac{\beta^2}{4a}\right]$$

$\Rightarrow \beta^2-y\beta-4ax=0$ which is a quadratic in β.

$\therefore$ Envelope is $(-y)^2=4(-4ax)$

i.e., $y^2+16ax=0$ which is also a parabola.

Ex. 5. Find the envelope of the polar of the points on the ellipse $\dfrac{x^2}{h^2}+\dfrac{y^2}{k^2}=1$ w. r. t. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$ (C. H. 1966)

Let (α, β) be a point on the given ellipse. Then polar of (α, β) w. r. to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is

$$\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}=1 \qquad \cdots \quad (1)$$

Also, $\because$ (α, β) lies on $\dfrac{x^2}{h^2}+\dfrac{y^2}{k^2}=1$

$$\frac{\alpha^2}{h^2}+\frac{\beta^2}{k^2}=1 \qquad \cdots \quad (2)$$

Differentiating (1) and (2) w. r. to α

$$\frac{x}{a^2}+\frac{y}{b^2}\frac{d\beta}{d\alpha}=0$$

$$\text{and} \quad \frac{2\alpha}{h^2}+\frac{2\beta}{k^2}\frac{d\beta}{d\alpha}=0.$$

Eliminating $\frac{d\beta}{d\alpha}$ from these two equations we get

$$\frac{\frac{x}{a^2}}{\frac{\alpha}{h^2}}=\frac{\frac{y}{b^2}}{\frac{\beta}{k^2}}$$

$$\text{or,}\quad \frac{\frac{x\alpha}{a^2}}{\frac{\alpha^2}{h^2}}=\frac{\frac{y\beta}{b^2}}{\frac{\beta^2}{k^2}}=\frac{\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}}{\frac{\alpha^2}{h^2}+\frac{\beta^2}{k^2}}=1$$

$$\therefore\quad \alpha=\frac{xh^2}{a^2} \text{ and } \beta=\frac{yk^2}{b^2}.$$

$\therefore$ Putting in (1), the required envelope is

$$\frac{x^2h^2}{a^4}+\frac{y^2k^2}{b^4}=1.$$

12.8. Pedal curve.

The locus of the foot of the perpendicular drawn from a given point on the tangent to any point of a curve is called the pedal of the curve with respect to the given point.

If there are series of curves denoted by Γ, Γ_1, Γ_2, $\Gamma_3\cdots\Gamma_n$ such that each is a pedal to the curve immediately preceding it, then Γ_1 is called the first positive pedal of Γ, Γ_2 is is called the second positive pedal of Γ and so on. Thus Γ_n is called the nth positive pedal of Γ. Again if Γ_n be regarded as the original curve, then Γ_{n-1} is called the first negative pedal of Γ_n, Γ_{n-2} is called the second negative pedal of Γ_n and so on.

12.9. How to find the pedal with regard to the origin of any curve whose cartesian equation is given.

Let the equation of the curve be

$$f(x, y)=0$$

and let the equation to the tangent at any point $P(x, y)$ on the curve be

$$x \cos\alpha+y \sin\alpha=p.$$

$\Rightarrow$ Condition of tangency is of the form

$$\phi(p, \alpha)=0 \quad \cdots \quad \cdots \quad (1)$$

where (p, α) are the polar coordinates of the foot N of the perpendicular on the tangent.

If (r, θ) be the polar coordinates of N then replacing (p, α) in (1) by (r, θ), we find the pedal curve which is the locus of N in polar form.

Alternative method

Let the equation of the curve be $f(x, y)=0 \quad \cdots \quad (1)$

The equation to the tangent at any point $P(x, y)$ on the curve is

$$Y-y=\frac{dy}{dx}(X-x). \quad \cdots \quad \cdots \quad (2)$$

Also, equation of the perpendicular from the origin on the tangent (2) is

$$Y=-\frac{1}{\frac{dy}{dx}}X$$

$$\Rightarrow \quad X+Y\frac{dy}{dx}=0 \quad \cdots \quad \cdots \quad (3)$$

Since, the foot of the perpendicular N is the intersection of (2) and (3), its locus which is the pedal curve, is obtained by eliminating (x, y) from (1), (2) and (3).

Ex. 1. Show that the first positive pedal of the parabola $y^2=4ax$ with regard to its vertex is the curve $x(x^2+y^2)+ay^2=0$ (C. H. 1965 old)

Let $x\cos\alpha+y\sin\alpha=p \quad \cdots \quad \cdots \quad (1)$
touches the parabola at (x_1, y_1).

Now, equation of the tangent to the parabola at (x_1, y_1) is

$$yy_1=2a(x+x_1)$$

$$\text{or,} \quad 2ax-yy_1+2ax_1=0 \quad \cdots \quad \cdots \quad (2)$$

Then (1) and (2) must be identical.

$\therefore$ Comparing co-efficient

$$\frac{2a}{\cos\alpha}=\frac{-y_1}{\sin\alpha}=\frac{-2ax_1}{p}$$

$\therefore$ $y_1=-2a\tan\alpha$ and $x_1=-p\sec\alpha$.

Since, (x_1, y_1) lies on $y^2=4ax$

$$y_1{}^2=4ax_1$$

$\therefore$ $4a^2\tan^2\alpha=-4ap\sec\alpha$

Hence, the condition of tangency is

$$a\tan^2\alpha+p\sec\alpha=0$$

Now, replacing (p, α) by the polar co-ordinates (r, θ) the required pedal is

$$a\tan^2\theta+r\sec\theta=0$$

or, $r+a\tan\theta\sin\theta=0.$

Transferring into cartesian form the equation becomes

$$r+a\frac{y}{x}\cdot\frac{y}{r}=0$$

or, $r^2x+ay^2=0$

or, $x(x^2+y^2)+ay^2=0.$

Alternative method

Equation to the parabola is $y^2=4ax$ ... (1)

and equation to the tangent to the parabola for all values of m is $y=mx+\frac{a}{m}$ ... ... (2)

Then the equation of the perpendicular from the origin on (2) is

$$y=-\frac{1}{m}x \quad \cdots \quad \cdots \quad (3)$$

Now, the intersection of (2) and (3) is the foot of the perpendicular on the tangent. Hence, the locus of the foot of the perpendicular which is the pedal curve, is obtained by eliminating m from (2) and (3).

From (3) we get $m=-\frac{x}{y}$.

Putting this value of m in (2), we get

$$y=-\frac{x^2}{y}-\frac{ay}{x}$$

or, $xy^2=-x^3-ay^2$

or, $x(x^2+y^2)+ay^2=0$ which is the required pedal.

Ex. 2. Find the first positive pedal of the rectangular hyperbola $x^2-y^2=a^2$ with respect to the centre.

Show that the pedal is the lemniscate $r^2=a^2\cos 2\theta$.

(C. H. 1970)

Let $(a\sec\phi, a\tan\phi)$ be any point on the rectangular hyperbola.

Then equation of the tangent at ϕ, is

$$x\sec\phi-y\tan\phi=a \quad \cdots \quad \cdots \quad (1)$$

Equation of the perpendicular from the origin on (1) is

$$x\tan\phi+y\sec\phi=0 \quad \cdots \quad \cdots \quad (2)$$

Since, intersection of (1) and (2) is the foot of the perpendicular, the required pedal is obtained by eliminating ϕ from (1) and (2).

From (2) $\dfrac{\sec\phi}{x}=\dfrac{\tan\phi}{-y}=\dfrac{1}{\sqrt{x^2-y^2}}$

$\therefore\quad \sec\phi=\dfrac{x}{\sqrt{x^2-y^2}}$ and $\tan\phi=-\dfrac{y}{\sqrt{x^2-y^2}}$.

Putting these values of $\sec\phi$ and $\tan\phi$ in (1) the required pedal is

$$\frac{x^2}{\sqrt{x^2-y^2}}+\frac{y^2}{\sqrt{x^2-y^2}}=a$$

or, $(x^2+y^2)^2=a^2(x^2-y^2)$

Transferring in polar coordinates this equation becomes

$$r^4=a^2(r^2\cos^2\theta-r^2\sin^2\theta)$$

or, $r^2=a^2\cos 2\theta$. which is a lemniscate.

12.10. From the definition of the pedal of a curve it follows that every circle described with any radius vector as diameter touches the pedal. So the envelope of such circles will be the pedal of the curve.

Ex. 1. Find the pedal of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to the centre. (C. H. 1958)

Let C be the centre and $P(a\cos\phi, b\sin\phi)$ be any point on the ellipse.

Then equation to the circle with CP as diameter is

$$x^2+y^2-ax\cos\phi-by\sin\phi=0 \qquad \cdots \quad (1)$$

Differentiating (1) with respect to ϕ

$$ax\sin\phi-by\cos\phi=0$$

$$\Rightarrow \quad \tan\phi=\frac{by}{ax}.$$

$$\therefore \quad \sin\phi=\frac{by}{\sqrt{a^2x^2+b^2y^2}}, \quad \cos\phi=\frac{ax}{\sqrt{a^2x^2+b^2y^2}}.$$

Putting these values of $\sin\phi$ and $\cos\phi$ in (1) the envelope of the circles i.e., the required pedal is

$$x^2+y^2=\frac{a^2x^2}{\sqrt{a^2x^2+b^2y^2}}+\frac{b^2y^2}{\sqrt{a^2x^2+b^2y^2}}$$

$$=\sqrt{a^2x^2+b^2y^2}$$

$\Rightarrow \quad (x^2+y^2)^2-(a^2x^2+b^2y^2)=0$ is the required pedal to the ellipse with respect to its centre.

Ex. 2. Find the pedal with respect to the pole of the curve $r^2=a^2\cos 2\theta$.

Let (ρ, α) be any point on the curve.

Then $\quad \rho^2=a^2\cos 2\alpha \qquad \cdots \qquad \cdots \quad (1)$

Now equation of the circle with radius vector ρ as diameter is

$$r=\rho\cos(\theta-\alpha) \qquad \cdots \qquad \cdots \quad (2)$$

Differentiating (1) and (2) with respect to ρ

$$2\rho = -2a^2 \sin 2\alpha \frac{d\alpha}{d\rho}$$

and $0 = \cos(\theta - \alpha) + \rho \sin(\theta - \alpha)\dfrac{d\alpha}{d\rho}$.

Eliminating $\dfrac{d\alpha}{d\rho}$ from these two equations, we get

$$\frac{\rho}{\cos(\theta - \alpha)} = \frac{a^2 \sin 2\alpha}{\rho \sin(\theta - \alpha)}$$

or, $\rho^2 \sin(\theta - \alpha) = a^2 \cos(\theta - \alpha) \sin 2\alpha$

or, $a^2 \cos 2\alpha . \sin(\theta - \alpha) = a^2 \cos(\theta - \alpha) \sin 2\alpha$ from (1)

or, $\tan(\theta - \alpha) = \tan 2\alpha$

$\therefore \theta - \alpha = 2\alpha$ i.e., $\theta = 3\alpha$

So, from (2)

$r = \rho \cos 2\alpha = \rho . \rho^2 / a^2$

$\therefore \rho = r^{\frac{1}{3}} a^{\frac{2}{3}}$. ... ... (3)

Again, from (1) $\rho^2 = a^2 \cos \frac{2}{3}\theta$ .. (4)

$\therefore$ From (3) and (4)

$$r^{\frac{2}{3}} a^{\frac{4}{3}} = a^2 \cos \tfrac{2}{3}\theta$$

or, $r^{\frac{2}{3}} = a^{\frac{2}{3}} \cos \frac{2}{3}\theta$ is the required pedal.

Exercise 12

1. Show that the envelope of the family of semicubical parabolas $y^2 - (x+\alpha)^3 = 0$ is the x axis.

2. Show that the envelope of the family of circles passing through the vertex of the parabola $y^2 = 4ax$ and the centres lying on it is the curve

$$y^2(2a+x) + x^3 = 0.$$

3. Prove that the envelope of the parabola

$\sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{y}{b}} = 1$ where the parameters are connected by the relation $ab = k^2$ is the hyperbola $16xy = k^2$.

4. Show that the envelope of the family of circles whose centre lie on the hyperbola $xy=c^2$ and which passes through its centre is the curve

$$(x^2+y^2)^2=16c^2xy.$$

5. Show that the envelope of the straight line

$$x/a+y/b=1$$

where the parameters a and b are connected by the relation $a^2+b^2=c^2$ is the astroid $x^{2/3}+y^{2/3}=c^{2/3}$.

6. Show that the envelope of the straight line

$$x/l+y/m=1$$

where the parameters l and m are connected by the relation $lm=c^2$(c being, constant) is the hyperbola $4xy=c^2$

7. Prove that the envelope of the straight line

$$\frac{x}{h}+\frac{y}{k}=1$$

where the parameters h, k are connected by the relation $h/a+k/b=1$, (a and b being constants) is the parabola

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$$

Show that the envelope of the family of ellipses

$x^2/a^2+y^2/b^2=1$ where the parameters a and b are connected by the relation $a+b=k$, (k being a constant) is the astroid

$$x^{2/3}+y^{2/3}=k^{2/3}.$$

9. Prove that the envelope of the circles

$$(x-\alpha)^2+(y-\beta)^2=\alpha^2+\beta^2$$

whose centres lie on the parabola $y^2=4ax$, (α, β being parameters,) is $x(x^2+y^2-2ax)=0$.

10. Show that the envelope of the curves

$$\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1$$

whose parameters α, β lie on the ellipse $x^2/a^2+y^2/b^2=1$ is the ellipse itself.

11. Show that the envelope of the straight line joining the extremities of a pair of conjugate diameters of the ellipse $x^2/a^2+y^2/b^2=1$ is another ellipse $x^2/a^2+y^2/b^2=1/2$.

12. Prove that the envelope of the circles described on the double ordinates of the parabola $y^2=4ax$ as diameter is the parabola $y^2=4a(x+a)$.

13. Prove that the envelope of the chord of the parabola $y^2=4ax$ which subtends a right angle at the focus is the ellipse $(x-3a)^2+2y^2=8a^2$.

14. Show that the envelope of the family of lines through a point P perpendicular to the polar of P with respect to the parabola $y^2=4ax$ and passing through a fixed point (h, k) is the parabola $(x-2a+h)^2+4ky=0$,

15. Given that the envelope of the family of straight lines whose intercepts on the two axis are the variable parameters a and b is the astroid $x^{2/3}+y^{2/3}=c^{2/3}$. Prove that $a^2+b^2=c^2$

16. Prove that the envelope of the circles described on the double ordinates of the ellipse $x^2/a^2+y^2/b^2=1$ as diameters is another ellipse

$$\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1.$$

17. Show that the radius of curvature of the envelope of the line $x\cos\theta+y\sin\theta=f(\theta)$ is $f(\theta)+f''(\theta)$.

18. Prove that the evolute of the astroid

$$x^{2/3}+y^{2/3}=a^{2/3}$$

is the curve $(x+y)^{2/3}+(x-y)^{2/3}=2a^{2/3}$.

19. Prove that the evolute of the ellipse $x^2/a^2+y^2/b^2=1$ isthe curve $(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$.

20. Show that the envelope of straight lines drawn at right angles to the radius vectors of the cardioide $r=a(1+\cos\theta)$ is the curve $r=2a\cos\theta$.

21. Show that the equation of the pedal of the parabola

$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ with respect to origin is the curve

$(x^2+y^2)(ax+by)=abxy$.

22. Show that the pedal of the cardioide

$$r=a(1+\cos\theta)$$

with respect to its pole is $r=2a\cos^3\left(\frac{\theta}{3}\right)$.

23. Show that the pedal of the circle $r-2a\cos\theta$ with respect to its origin is the cardioide

$$r=a(1+\cos\theta),$$

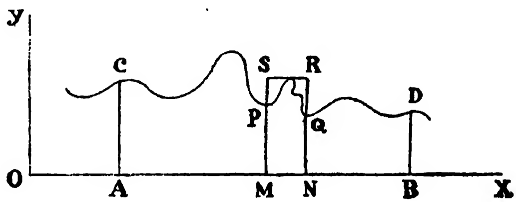
CHAPTER XIII

AREAS OF PLANE REGIONS

13.1. Areas in cartesian coordinates.

Let the curve $y=f(x)$ be single valued, finite and continuous in the interval (a, b). Then the area bounded by the curve $y=f(x)$, the x-axis and two ordinates at $x=a$ and $x=b$ is

$$\int_a^b y\,dx.$$



Let the ordinates of curve $y=f(x)$ at $x=a$ and b be AC and BD so that $OA=a$ and $OB=b$. Let $P(x, y)$ be any point on the curve and the corresponding ordinate at this point be PM, so that $OM=x$. Let the area bounded by the curve, the x-axis and the ordinates at A and M be denoted by A.

As the point A is fixed and M is variable depending on x, the area A is a function of x. $\therefore$ As x is increased by an amount δx, let the area be increased by an amount δA, the area enclosed by the ordinates PM and QN. Let $f(x_1)$ and $f(x_2)$ be the greatest and the least ordinates in δx. Then clearly

$$f(x_2)\delta x<\delta A<f(x_1)\delta x$$

$$\Rightarrow \quad f(x_2)<\frac{\delta A}{\delta x}< f(x_1) \qquad \cdots \qquad \cdots \qquad (1)$$

Now, since $f(x)$ is continuous at x, as $Q \to P$ i.e., $\delta x \to 0$, both $f(x_1)$ and $f(x_2)$ approach $f(x)$, and $\frac{\delta A}{\delta x}$ tends to $\frac{dA}{dx}$.

Hence, as the relation (1) should hold true always, proceeding to the limit, we get

$$\frac{dA}{dx} = f(x).$$

$$\Rightarrow \quad A = \int f(x)dx + c$$

$= F(x) + c$ where c is an arbitrary constant.

When $x = a$, PM coincides with CA and so area becomes zero. When $x = b$, PM coincides with BD and the area becomes the required area A_1 (say)

$$\therefore \quad 0 = F(a) + c$$

$$A_1 = F(b) + c$$

$$\Rightarrow \quad A_1 = F(b) - F(a)$$

$$= \int_a^b f(x)\, dx$$

$$= \int_a^b y\, dx.$$

Cor. 1. In the similar way, it can be shown that the area included between the curve $x = f(y)$, the y axis and the two abscissae $y = c$, $y = d$ is

$$\int_c^d f(y)\, dy \quad \text{i.e.,} \quad \int_c^d x\, dy$$

Cor. 2. If the equation be given in the parametric form $x = f(\theta)$, $y = \phi(\theta)$, then the area with proper limits θ_1 and θ_2 is $\int_{\theta_1}^{\theta_2} f(\theta)\phi'(\theta)\, d\theta$.

Cor. 3. If the axes be not rectangular but ω be the inclination between them, the area under consideration would be

$$\sin\omega\int_a^b f(x)\,dx \text{ and } \sin\omega\int_a^b f(y)\,dy \quad \text{respectively.}$$

Ex. 1. Find the total area of the circle $x^2+y^2=a^2$.

The area bounded by the curve, the x axis and the ordinates $x=0$ and $x=a$ is the area of the quadrant of the circle. Hence, the required area

$$=4\int_0^a y\,dx=4\int_0^a \sqrt{a^2-x^2}\,dx$$ [$\because$ in the portion considered y is positive.]

$$=4\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$=\frac{4a^2}{2}\cdot\frac{\pi}{2}=\pi a^2 \text{ square units.}$$

Otherwise. Let $x=a\cos\theta$, $y=a\sin\theta$ be the parametric equation of the circle. Then the required area

$$=4\int_0^a y\,dx=4\int_{\pi/2}^0 a\sin\theta(-a\sin\theta)d\theta$$

$$=4a^2\int_0^{\pi/2}\sin^2\theta\,d\theta=4a^2\cdot\frac{\pi}{4}=\pi a^2 \text{ square units}$$

Ex. 2. Find the total area of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The area bounded by the curve, the x axis and the

ordinates at $x=0$ and $x=a$ is the area of the quadrant of the ellipse. Hence, the required area of the ellipse

$$=4\int_0^a y\,dx$$

$$=4\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,.b\,dx \qquad [\because \text{ In the portion considered } y \text{ is positive }]$$

$$=\frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx.$$

$$=\frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$=\frac{4b}{a}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2}=\pi ab \text{ square units.}$$

Otherwise. Let $x=a\cos\theta$, $y=b\sin\theta$ be the parametric equation of the ellipse. Then the required area

$$=4\int_0^a y\,dx=4\int_{\pi/2}^0 b\sin\theta(-a\sin\theta)d\theta$$

$$=4ab\int^{\pi/2}\sin^2\theta\,d\theta=4ab\,.\frac{\pi}{4}=\pi ab \text{ square units.}$$

Ex. 3. Find the area included between the parabola $y^2=4ax$ and the double ordinate at $x=x_1$.

The area included between the curve, the x-axis and the ordinates at $x=0$ and $x=x_1$ is $\frac{1}{2}$ of the required area.

$$\therefore \quad \text{Required area} = 2\int_0^{x_1} y\,dx$$

$$=2\int_0^{x_1}\sqrt{4ax}\,dx \qquad [\because \text{ In the portion considered } y \text{ is positive.}]$$

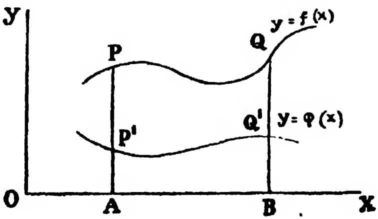
$$=4\sqrt{a}\left[x^{\frac{3}{2}}\Big/\frac{3}{2}\right]_0^{x_1}=\frac{8}{3}a^{\frac{1}{2}}x_1^{\frac{3}{2}}.$$

$$=\frac{8}{3}(ax_1)^{\frac{1}{2}}.\,x_1$$

$$=\frac{8}{3}\,\frac{y_1}{2}\,.\,x_1=\frac{4}{3}x_1y_1 \qquad \left[\because \quad y_1=\sqrt{4ax_1}\right]$$

13.2. Area included between two curves.

Let $y=f(x)$ and $y=\phi(x)$ be two curves and the ordinates at A and B on the x axis cut the curves at P, Q and P', Q' respectively. Let $OA=a$ and $OB=b$.



$$\text{Then, area } PABQ=\int_a^b f(x)\,dx$$

$$\text{and} \quad \text{area } P'ABQ'=\int^b \phi(x)\,dx.$$

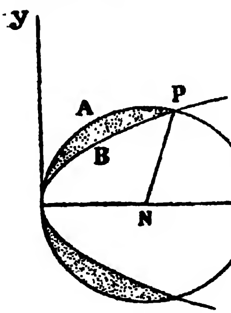
$\therefore$ Area $PP'Q'Q=$ Area $PABQ-$ Area $P'ABQ'$

$$=\int_a^b f(x)\,dx-\int_a^b \phi(x)\,dx$$

$$=\int_a^b \{f(x)-\phi(x)\}dx$$

$$=\int_a^b (y_1-y_2)\,dx.$$

Ex. 1. Find the area included between the circle $x^2+y^2=2ax$ and the parabola $y^2=ax$.



The two curves intersect at (0, 0), (a, a) and $(a, -a)$.

The required area

$=2\times$area of $OAPBO$ (from symmetry)

$$=2\left\{\int_0^a y_1dx-\int_0^a y_2dx\right\}$$

$$=2\left\{\int_0^a \sqrt{2ax-x^2}\,dx-\int_0^a \sqrt{ax}\,dx\right\} \quad \cdots \quad (1)$$

1st Integral $=\int_0^a \sqrt{a^2-(a-x)^2}$

$$=\int_{\pi/2}^0 a\cos\theta(-a\cos\theta)d\theta \quad [\text{ Putting } a-x=a\sin\theta.]$$

$$=a^2\int_0^{\pi/2} \cos^2\theta\,d\theta=a^2\frac{\pi}{4}.$$

2nd integral $= \sqrt{a}\int_0^{a} x^{\frac{1}{2}}\,dx = \sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{a} = \frac{2}{3}a^2$

$\therefore$ from (1), the required area $= 2\left[\frac{\pi a^2}{4} - \frac{2a^2}{3}\right]$

$$= 2a^2\left(\frac{\pi}{4} - \frac{2}{3}\right).$$

Ex. 2. Find the area of the ellipse

$$\mathbf{ax^2 + 2hxy + by^2 = 1.}$$

Writing the equation as a quadratic in y

$$by^2 + 2hxy + (ax^2 - 1) = 0$$

$$\therefore \quad y = \frac{-2hx \pm \sqrt{4h^2x^2 - 4b(ax^2 - 1)}}{2b}$$

$$= \frac{-hx \pm \sqrt{b - (ab - h^2)x^2}}{b} \quad \left[\because \text{ It is an ellipse } ab - h^2 \text{ is positive}\right]$$

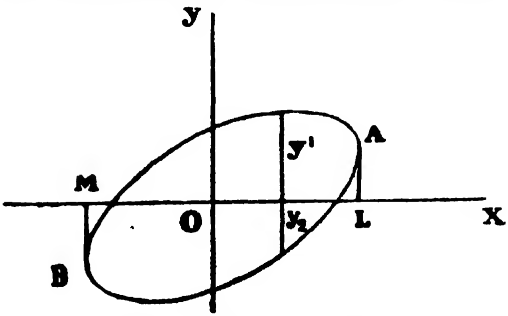
Let y_1 and y_2 be the two values of y.

Then $y_1 - y_2 = \left\{\frac{-hx + \sqrt{b - (ab - h^2)x^2}}{b}\right\}$

$$- \left\{\frac{-hx - \sqrt{b - (ab - h^2)x^2}}{b}\right\}$$

$$= \frac{2}{b}\sqrt{b - (ab - h^2)x^2}.$$

Let the ordinates at the extreme points A and B meet



the x axis at L and M. Then $OL = OM$.

$\therefore$ the extreme values of x are given by $y_1 - y_2 = 0$.

$$\Rightarrow b-(ab-h^2)x^2=0$$

$$\Rightarrow x=\pm\sqrt{\frac{b}{ab-h^2}}=\pm\alpha \text{ (say)}$$

$\therefore$ Required area

$$\int_{-\alpha}^{\alpha}(y_1-y_2)dx=\frac{2}{b}\int_{-\alpha}^{\alpha}\sqrt{b-(ab-h^2)x^2}\,dx \qquad \cdots \quad (1)$$

[Put $\sqrt{(ab-h^2)}x=\sqrt{b}\sin\theta$.

When $x=\alpha=\sqrt{\dfrac{b}{ab-h^2}}$, $\sin\theta=1 \quad \therefore \quad \theta=\pi/2$

When $x=-\alpha=-\sqrt{\dfrac{b}{ab-h^2}}$, $\sin\theta=-1 \quad \therefore \quad \theta=-\pi/2$

$\therefore$ Also $\sqrt{ab-h^2}\,dx=\sqrt{b}\cos\theta\,d\theta$.

$\therefore$ $dx=\sqrt{\dfrac{b}{ab-h^2}}\cos\theta\cdot d\theta$]

So, from (1), the required area

$$=\frac{2}{b}\cdot\sqrt{\frac{b}{ab-h^2}}\cdot\sqrt{b}\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta.$$

$$=\frac{2}{\sqrt{ab-h^2}}\cdot 2\int_{0}^{\pi/2}\cos^2 d\theta.$$

$$=\frac{4}{\sqrt{ab-h^2}}\times\frac{\pi}{4}=\frac{\pi}{\sqrt{ab-h^2}}.$$

13.3. On the Sign of an Area.

On the formula

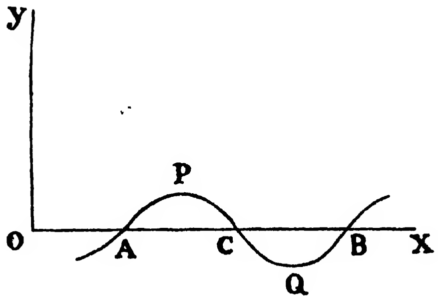
$$A=\int_a^b y\,dx.$$

If the variable ordinate y is positive and $b>a$, then clearly A is positive.

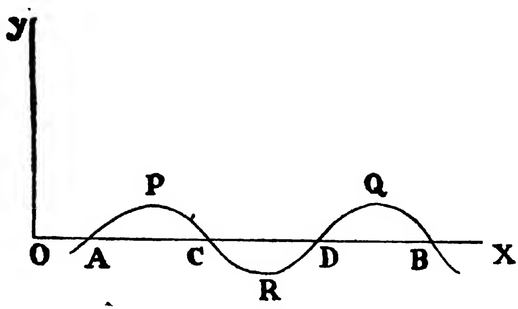
If the variable ordinate y is negative and $b>a$ or, if y is positive and $b<a$, then A is negative.

So, difficulty arises regarding sign of an area if in the same region, the variable ordinate y is somewhere positive and somewhere negative, remaining always greater than a; or if in the same region the variable ordinate y remains always positive but b remaining always less than a.

Let the curve cut the x-axis at A, B, C, D whose abscissae are respectively a, b, c and d etc.



(i)



(ii)

In the first diagram, the ordinate between A and C is positive and $c>a$, but the ordinate between C and B is negative and $b>c$.

So, area APC is positive while area CQB is negative.

$$\therefore \quad \text{Total area} = \int_a^b y\,dx = \int_a^c y\,dx - \int_c^b y\,dx \quad \cdots \quad (1)$$

In the second diagram, the ordinates between A and C also between D and B are positive and $c>a$, $b>d$; but the ordinate between C and D is negative and $d>c$.

So, here, areas APC and DQB are positive while the area CRD is negative.

$$\therefore \quad \text{Total area} = \int_a^b y\,dx = \int_a^c y\,dx + \int_d^b y\,dx - \int_c^d y\,dx \quad \cdots \quad (2)$$

From (1) and (2), we may now state the following two rules :—

(1) While going round the boundary, if the region whose area is asked for lies to the right, then the area is positive ; on the other hand, if the region lies always to the left of the boundary, then the area is negative.

(2) If the curve cuts the x-axis at one or more points, then

$$\int^{v} f(x)dx$$

represents the difference of the area on the right over that on the left which may be positive or negative or even zero if the magnitudes of the two areas are equal.

If our object is to find the numerical value of the total area then we are to consider the magnitude of the sub-areas.

So in Fig. (i)

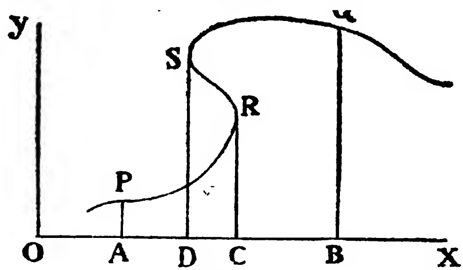
$$A = \int_a^b y\,dx = \left|\int_a^c y\,dx\right| + \left|-\int_c^b y\,dx\right|$$

$$= \left|\int_a^c y\,dx\right| + \left|\int_c^b y\,dx\right|$$

Similarly, in Fig. (ii)

$$A=\int_a^b ydx=\left|\int_a^c ydx\right|+\left|\int_d^b ydx\right|+\left|\int_c^d ydx\right|$$

Again, if the curve is of the form as in Fig. (iii), the area between AC and DB being right side of the curve PR and



(iii)

SQ are clearly positive. The area between DC being left side of the curve RS is negative.

$\therefore$ Area $PACR$ is positive, area $SDCR$ is negative and area $SDBQ$ is positive.

$$\therefore \quad \int_a^b ydx=\text{Area } PACR+\text{Area } SDBQ-\text{Area } SDCR$$

$$=\int_a^c ydx+\int_d^b ydx-\int_d^c ydx.$$

Thus, the area between the curve, the x-axis and the ordinates at $x=a$ and $x=b$ is the algebraic sum of the three areas with their proper signs.

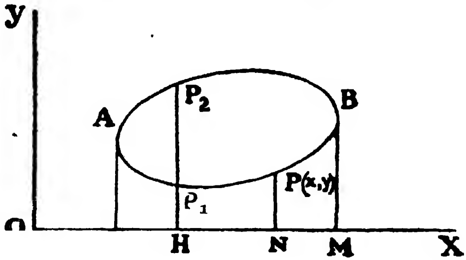
13.4. Area bounded by a closed curve

$x=f(t)$, $y=g(t)$ is

$$\frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt.$$

where the parameter t is such that as it increases from t_1 to t_2 in the anti-clockwise sense, the whole curve is described.

Let $P(x, y)$ be any position on the closed curve whose parametric equations are $x=f(t)$, $y=g(t)$ where t is a vari-



(iv)

able parameter. Let the ordinates at $x=a$ and $x=b$ touches the curve at A and B respectively.

Suppose, the curve does not intersect at any point within A and B.

Then, as the curve is a closed curve, every ordinate between $x=a$ and $x=b$ will cut the curve in two points at P_1 and P_2, one on the lower part and the other on the upper part of the curve.

Also, suppose, as the parameter t increases in the anti-clockwise sense from t_1 at P to the value t_2, the whole curve is described.

Now the area of the closed curve

$$=\text{Area } ALMBP_2A-\text{Area } ALMBP_1A$$

$$=\int_a^b HP_2dx-\int_a^b HP_1dx.$$

$$=-\int^{a} HP_2dx-\int^{b} HP_1dx \qquad (1)$$

Let t_b and t_a denote the values of the parameter t at B and A respectively, so that as $P(x, y)$ describes the curve BP_2A, t increases from t_b to t_a.

$$\therefore \quad \int_b^a HP_2 dx = \int_{t_b}^{t_a} y\frac{dx}{dt}dt$$

Again, as $P(x, y)$ describes the curve from A to P, t increases from t_a to t_2 and as it describes the curve from P to B, t increases from t_1 to t_b.

$$\therefore \quad \int_a^b HP_1 dx = \int_a^{ON} HP_1 dx + \int_{ON}^b HP_1 dx$$

$$= \int_{t_a}^{t_2} y\frac{dx}{dt}dt + \int_{t_1}^{t_b} y\frac{dx}{dt}dt$$

So, from (1), the area of the closed region

$$= -\int_{t_b}^{t_a} y\frac{dx}{dt}dt - \int_{t_a}^{t_2} y\frac{dx}{dt}dt - \int_{t_1}^{t_b} y\frac{dx}{dt}dt$$

$$= -\left\{\int_{t_b}^{t_a} y\frac{dx}{dt}dt + \int_{t_a}^{t_2} y\frac{dx}{dt}dt\right\} + \int_{t_b}^{t_1} y\frac{dx}{dt}dt$$

$$= -\int_{t_b}^{t_2} y\frac{dx}{dt}dt + \int_{t_b}^{t_1} y\frac{dx}{dt}dt$$

$$= \int_{t_2}^{t_b} y\frac{dx}{dt}dt + \int_{t_b}^{t_1} y\frac{dx}{dt}dt$$

$$= \int_{t_2}^{t_1} y\frac{dx}{dt}dt$$

$$= -\int^{t_2} y\frac{dx}{dt}dt.$$

Similarly, by drawing ordinates on the y-axis, it can be shown that this area

$$= \int_{t_1}^{t_2} x\frac{dy}{dt}dt.$$

Hence, if A denotes the required area, then

$$2A = \int_{t_1}^{t_2} x\frac{dy}{dt}dt - \int_{t_1}^{t_2} y\frac{dx}{dt}dt$$

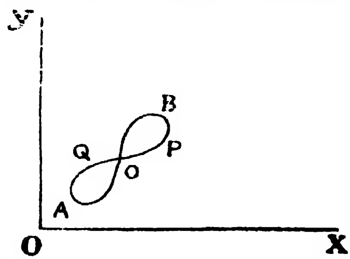
$$= \int_{t_1}^{t_2} \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

$$\therefore \quad A = \frac{1}{2}\int_{t_1}^{t_2} \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt. \qquad \ldots \quad (1)$$

Cor. 1. The formula (1) may conveniently be put in the form

$$A = \frac{1}{2}\int (xdy - ydx).$$

Cor. 2. If the curve intersect at a point within A and



B, then the difference of the areas of the two loops on either side of the point of intersection is

$$= \frac{1}{2}\int (xdy - ydx).$$

In the adjoining diagram let O be the point of intersection dividing the region into two loops. Let t vary from t_1 to t'_1 as the curve is described along OPB from O back to O. Also let t vary from t'_1 to t_2 as the curve is described along OAQ from O back to O.

Then area of the loop $OPBO = \frac{1}{2}\int_{t_1}^{t'_1}(xdy - ydx)$

and the area of the loop $OAQO = -\frac{1}{2}\int^{t_2}(xdy - ydx)$

$\therefore$ Difference of the two loops $=\frac{1}{2}\int_{t_1}^{t_2}(xdy - ydx)$

So, in such a case, the total area is obtained by taking the sum of the two areas obtained separately.

Ex. Find the area of the region enclosed by the circle $x = a \cos t$, $y = a \sin t$.

As the circle is a closed curve, the whole area is described as t varies from 0 to 2π.

Hene, $\quad x\dfrac{dy}{dt} = a \cos t \cdot a \cos t = a^2 \cos^2 t.$

and $\quad y\dfrac{dx}{dt} = a \sin t\ (-a \sin t) = -a^2 \sin^2 t.$

$$\therefore \quad A = \frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

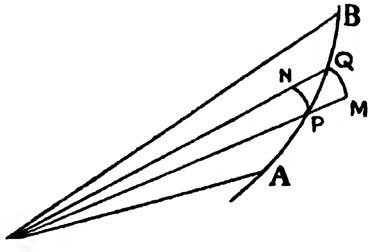
$$= \frac{1}{2}\int_0^{2\pi} a^2\, dt = \frac{1}{2}a^2 . 2\pi = \pi a^2.$$

Similarly, area enclosed by the ellipse
$x = a \cos t$, $y = b \sin t$ is πab.

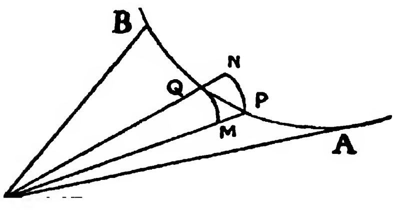
13.5. Area in polar coordinates

If $r=f(\theta)$ be the equation to a curve in polar coordinates, then the area of the region enclosed by the curve and the two radii vectors $\theta=\alpha$ and $\theta=\beta$ is (C.H. 1970)

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta.$$



(I)



(II)

Let OA and OB be two fixed radii vectors of the curve $r=f(\theta)$, so that $\angle AOX=\alpha$ and $\angle BOX=\beta$.

Let $P(r, \theta)$ be any point on the curve, so that the variable area AOP is denoted by A.

Let $Q(r+\delta r, \theta+\delta\theta)$ be a point on the curve very near to P so that as P moves to Q, the radius vector constantly increases as in Fig. (I) or constantly decreases as in Fig. (II). Let the area POQ is denoted by δA.

With centre at O and OP, OQ as radii two arcs PN and QM are drawn.

Then, clearly δA lies between the two sectorial areas OPN and OQM.

So in Fig. (I)

$$\tfrac{1}{2}r^2\delta\theta < \delta A < \tfrac{1}{2}(r+\delta r)^2\delta\theta$$

$$\Rightarrow \quad \frac{1}{2}r^2 < \frac{\delta A}{\delta\theta} < \frac{1}{2}(r+\delta r)^2$$

$$\Rightarrow \quad \frac{1}{2}\{f(\theta)\}^2 < \frac{\delta A}{\delta\theta} < \frac{1}{2}\{f(\theta+\delta\theta)^2\} \qquad \cdots \quad (1)$$

And in Figure II

$$\frac{1}{2}(r+\delta r)^2\delta\theta < \delta A < \frac{1}{2}r^2\delta\theta$$

$$\Rightarrow \quad \frac{1}{2}(r+\delta r)^2 < \frac{\delta A}{\delta\theta} < \frac{1}{2}r^2.$$

$$\Rightarrow \quad \frac{1}{2}f(\theta+\delta\theta)\}^2 < \frac{\delta A}{\delta\theta} < \frac{1}{2}\{f(\theta)\}^2 \qquad \cdots \quad (2)$$

Now as $Q \to P$, $\delta\theta \to 0$.

Also as $f(\theta)$ is continuous, $f(\theta+\delta\theta) \to f(\theta)$ as $\delta\theta \to 0$.

So, proceeding to the limit as $Q \to P$ both (1) and (2) takes the form

$$\frac{dA}{d\theta} = \frac{1}{2}\{f(\theta)\}^2 = \frac{1}{2}r^2.$$

$$\Rightarrow \quad A = \frac{1}{2}\int r^2\, d\theta + c \text{ where } c \text{ is an arbitrary constant.}$$

$$= \frac{1}{2}F(\theta) + c \quad \text{(say)} \quad \cdots \quad (3)$$

Let A_1 denotes the required sectorial area AOB.

Then from (3)

$$A_1 = \tfrac{1}{2}F(\beta) + c$$

and $O = \tfrac{1}{2}F(\alpha) + c.$

$$\therefore \quad A_1 = \tfrac{1}{2}\{F(\beta) - F(\alpha)\}$$

$$= \tfrac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta.$$

Cor. The area bounded by the two curves $r_1 = f(\theta)$ and $r_2 = g(\theta)$ and the two radii vectors $\theta = \alpha$ and $\theta = \beta$ is

$$\frac{1}{2}\int_{\alpha}^{\beta}(r_2^2 - r_1^2)\,d\theta.$$

Ex. Find the area included between the ellipses

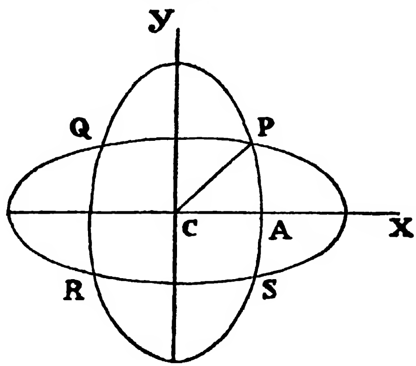
$$\mathbf{x^2+2y^2=a^2 \text{ and } 2x^2+y^2=a^2.}$$

Writing the equations in the standard form, we get

$$\frac{x^2}{a^2}+\frac{y^2}{\frac{1}{2}a^2}=1 \qquad \cdots \quad (1)$$

and
$$\frac{x^2}{\frac{1}{2}a^2}+\frac{y^2}{a^2}=1 \qquad \cdots \quad (2)$$

From (1) and (2), it is clear that the major axis of (1) and the minor axis of (2) lies along the x axis as in the diagram.



The point P, the intersection of (1) and (2) in the 1st quadrant is $(a/\sqrt{3}, a/\sqrt{3})$, obtained by solving (1) and (2).

$\therefore$ $\tan PCA = 1$

$\therefore$ $\angle PCA = \pi/4.$

Now, the arc AP being an arc of the ellipse (2), the sectorial area ACP is the area bounded by (2) and $\theta = 0$, $\theta = \pi/4$. And the required area $APQRSA = 8 \times$ Area ACP.

Writing $x=r\cos\theta$, $y=r\sin\theta$, the polar equation of (2) becomes

$$r^2=\frac{a^2}{2\cos^2\theta+\sin^2\theta}.$$

$$\therefore \text{ Area } APQRSA=8\times\tfrac{1}{2}\int_0^{\pi/4} r^2\,d\theta.$$

$$=4.\int_0^{\pi/4}\frac{a^2}{2\cos^2\theta+\sin^2\theta}d\theta.$$

$$=4a^2\int_0^{\pi/4}\frac{\sec^2\theta}{2+\tan^2\theta}d\theta.$$

$$=4a^2\int_0^1\frac{dz}{2+z^2}\quad [\text{Putting } z=\tan\theta]$$

$$=4a^2.\frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{z}{\sqrt{2}}\right]_0^1$$

$$=2\sqrt{2}\tan^{-1}\frac{1}{\sqrt{2}}$$

13.6. Area in Pedal form.

It $p=f(r)$ be the equation to a curve in pedal form, then then the area of the region enclosed by the curve and the two radii vectors $r=r_1$ and $r=r_2$ is (C.H. 1972)

$$\frac{1}{2}\int_{r_1}^{r_2}\frac{pr}{\sqrt{r^2-p^2}}dr.$$

Let ϕ be the angle between the radius vector and the tangent at any point $P(r,\ \theta)$ on the curve. Also let p be

the length of the perpendicular from the pole on the tangent at P. Then

$$p=r\sin\phi$$

$$\therefore \quad \frac{1}{p^2}=\frac{1}{r^2}\operatorname{cosec}^2\phi=\frac{1}{r^2}(1+\cot^2\phi)$$

$$=\frac{1}{r^2}\left\{1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\}$$

$$\Rightarrow \quad \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{1}{p^2}-\frac{1}{r^2}=\frac{r^2-p^2}{r^2p^2}$$

$$\therefore \quad \frac{d\theta}{dr}=\pm\frac{p}{r}\frac{1}{\sqrt{r^2-p^2}}.$$

Now, $\quad \text{area}=\frac{1}{2}\int r^2\,d\theta$

$$=\frac{1}{2}\int r^2\frac{d\theta}{dr}dr$$

$$=\frac{1}{2}\int_{r_1}^{r_2} r^2\cdot\frac{p}{r\sqrt{r^2-p^2}}dr$$

$$=\frac{1}{2}\int_{r_1}^{r_2}\frac{pr}{\sqrt{r^2-p^2}}dr$$

Ex. Find the sectorial area of the equi-angular spiral $p=r\sin\alpha$ enclosed by the curve and the radii vectors r_1 and r_2.

$$A=\frac{1}{2}\int_{r_1}^{r_2}\frac{pr\,dr}{\sqrt{r^2-p^2}}=\frac{1}{2}\int_{r_1}^{r_2}\frac{r\sin\alpha\,.\,r\,dr}{\sqrt{r^2-r^2\sin^2\alpha}}$$

$$=\frac{1}{2}\tan\alpha\int_{r_1}^{r_2}r\,dr=\frac{1}{2}\tan\alpha\left[\frac{r^2}{2}\right]_{r_1}^{r_2}$$

$$=\frac{1}{4}(r_2^2-r_1^2)\tan\alpha.$$

13.7. Area in Tangential Polar form.

If $p=f(\psi)$ be the equation to a curve in tangential polar form, then area enclosed by the curve and $\psi=\psi_1$ and ψ_2 is

$$=\frac{1}{2}\int_{\psi_1}^{\psi_2}p\left(p+\frac{d^2p}{d\psi^2}\right)d\psi.$$

Let $P(r, \theta)$ be any point on the curve $p=f(\psi)$ and $Q(r+\delta r, \theta+\delta\theta)$ be a point on the curve very near to P. Let p_1 be the length of the perpendicular from the pole on QP produced.

Let p be the length of the perpendicular from the pole on the tangent at P. Then, as $Q\to P$, chord $PQ\to\delta s$ and $p_1\to p$.

Then, the sectorial area $POQ=\frac{1}{2}\int p\, ds$.

$$=\frac{1}{2}\int p\rho\, d\psi. \qquad \left[\because\ \frac{ds}{d\psi}=\rho\right]$$

$$=\frac{1}{2}\int_{\psi_1}^{\psi_2} p\left(p+\frac{d^2p}{d\psi^2}\right)d\psi$$

Ex. Find the area enclosed by the curve

$$\mathbf{p=a(1+\sin\psi)}$$

Here $\frac{dp}{d\psi}=a\cos\psi$, $\quad\therefore\ \frac{d^2p}{d\psi^2}=-a\sin\psi$

So $p+\frac{d^2p}{d\psi^2}=a$

If ψ varies from O to 2π, the whole area is described.

$\therefore$ The area

$$=\frac{1}{2}\int_0^{2\pi} p\left(p+\frac{d^2p}{d\psi^2}\right)d\psi$$

$$=\frac{1}{2}\int_0^{2\pi} a(1+\sin\psi).\, a\, d\psi$$

$$=\frac{1}{2}a^2\Big[\psi-\cos\psi\Big]_0^{2\pi}=\frac{1}{2}a^2(2\pi)=\pi a^2$$

Exercise 13

1. Find the area bounded by the parabola $y=\frac{1}{4}x^2+1$ and its latus rectum. Ans. $\frac{8}{3}$.

2. Show that the area bounded by the semicubical parabola $y^2=cx^3$ and a double ordinate PQ, is $\frac{4}{5}PQ$ times the abscissa of P and Q.

3. Find the area bounded by the parabola

$$x^2+y^2+a^2-2xy-2ax-2ay=0$$

and the coordinate axes. Ans. $\frac{1}{3}a^2$.

4. Show that the area included between the circle $x^2+y^2=a^2$ and the astroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ is $\frac{5}{8}\pi a^2$.

5. Find the area of the hypo-cycloid

$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$$

and hence deduce that the area of the evolute

$$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$$

is $\frac{3}{8}\pi(a^2-b^2)^2/(ab)$. Ans. $\frac{3}{8}\pi ab$.

6. Find the area bounded by the parabolas $y^2=4x$ and $x^2=4y$ and show that the two curves divide the square bounded by $x=0$, $x=4$, $y=0$, $y=4$ into three equal areas. Ans. $\frac{16}{3}$.

7. Show that the area enclosed by the curve

$$a^4y^2=x^5(2a-x)$$

is $\frac{5}{4}\pi a^2$.

8. Show that the area enclosed between one arc of the cycloid $x=a(\theta\pm\sin\theta)$, $y=a(1-\cos\theta)$, and its base is $3\pi a^2$.

9. Find the asymptote of the curve

$$y^2(a+x)=(a-x)^2$$

Show that the area included between the curve and its asymptote is $3\pi a^2$.

10. If A be the vertex, O the centre and $P(x, y)$ any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, prove that $x=a\cosh\frac{2S}{ab}$, $y=b\sinh\frac{2S}{ab}$ where S is the sectorial area OPA. (Gauhati. 70)

11. Show that the area common to the two ellipses $x=a\cos\theta$, $y=b\sin\theta$ and $x=b\cos\theta$, $y=a\sin\theta$ $(a>b)$ is $2ab\tan^{-1}\frac{2ab}{a^2-b^2}$.

12. Show that for the curve

$$x=a(1-t^2),\ y=at(1-t^2),\ (-1\leqslant t\leqslant 1)$$

a single loop exists between $x=0$ and $x=a(>0)$ and that area of the loop is

13. Show that $x+y+a=0$ is an symptote to the folium $x^3+y^3=3axy$

Show that the area of the loop of the folium is $\frac{3}{2}a^2$ which is also the area between the folium and its asymptote.

14. Show that the area of the ellipse

$l/r=1+e\cos\theta$ is $\pi l^2/(1-e^2)^{\frac{3}{2}}$.

15. Prove that the area included between the folium $x^3+y^3=3axy$ and its asymptote is three times the area of one of the loops of $r^2=a^2\cos 2\theta$.

16. Show that $r^2(a^2\sin^2\theta+b^2\cos^2\theta)=a^2b^2$ represents the equation to an ellipse and that its area $=\pi ab$.

17. Find the whole area of ;

(i) $r=a+b\cos\theta$ $(b>a)$ Ans : $\pi\left(a^2+\dfrac{b^2}{2}\right)$

(ii) $r^2=a^2\cos^2\theta+b^2\sin^2\theta$. $\dfrac{\pi}{2}(a^2+b^2)$.

18. Show that the area enclosed by the curves

$r=2a(1+\cos\theta)$ and $r=\dfrac{2a}{1+\cos\theta}$ is $\dfrac{a^2}{3}\left(9\pi-16\right)$. (C. H. 1965)

19. Show that area of the curve

$$x=a\cos\theta+b\sin\theta+c,$$
$$y=a'\cos\theta+b'\sin\theta+c'$$

is $\pi(ab'-a'b)$.

20. Show that the curve $x=\dfrac{1}{e^t}(t-1)$, $y=tx$ has a loop of area $=\dfrac{1}{e}$.

21. Show that area of the portion included between the two cardioides $r=a(1-\cos\theta)$, $r=a(1+\cos\theta)$ is $\dfrac{a^2}{2}(3\pi-8)$. (C. H. 1966)

22. Show that the curve $x^5+y^5=5ax^2y^2$ consists of a single loop of area equal to $\frac{5}{2}a^2$.

23. Show that the curve

$$x=\frac{1-t^2}{1+t^2},\ y=\frac{t-t^3}{1+t^2}\ \ (-1\leq t\leq 1)$$

consists of a single loop of area $(2-\frac{1}{2}\pi)$.

CHAPTER—XIV

LENGTHS OF PLANE CURVES

14.1. Rectification of a Curve

If a curve $y=f(x)$ be continuous from P to Q and if this interval be divided in any manner by the points

$$A=P, P_1, P_2 \cdots\cdots P_{r-1}, P_r, \cdots\cdots, P_{n-1}, P_n=B$$

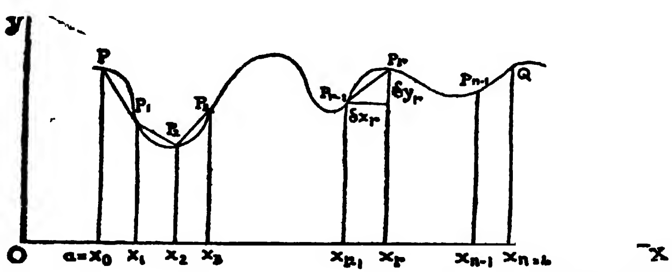
all lying on the curve, then the length of the sum of these broken lines,

$$AP_1+P_1P_2+P_2P_3+\cdots+P_{r-1}\,P_r+\cdots\cdots+P_{n-1}\,B$$

as the number of subdivisions tends to infinity in such a way that the greatest of these broken lines tends to zero, if exist, is defined to be the length of the curve AB and in such a case arc AB is said to be rectified between A and B.

14.2. If the derivative $f'(x)$ of a funtion $f(x)$ is continuous, the curve $y=f(x)$ is rectifiable and its length between $x=a$ and $x=b$ $(b>a)$ is given by (C.H. 1971)

$$s=\int^{b}[1+\{f'(x)\}^2]^{\frac{1}{2}}\,dx$$



Let $D\{a=x_0, x_1, x_2 \cdots\cdots x_{r-1}, x_r \cdots\cdots x_{n-1}, x_n=b\}$ be any mode of division of the interval (a, b) so that

$$(x_r-x_{r-1})=\delta x_r$$

Let the ordinates at these points be drawn to cut the curve at $P, P_1, P_2, P_3 \cdots\cdots P_{r-1}, P_r \cdots\cdots P_{n-1}, Q$.

Now, the length of the chord $P_{r-1}\, P_r$

$$= \sqrt{(\delta x_r)^2 + (\delta y_r)^2}$$

$$= \sqrt{1+\left(\frac{\delta y_r}{\delta x_r}\right)^2}\,\delta x_r \qquad \cdots \qquad \cdots \qquad (1)$$

As the curve $y=f(x)$ is continuous and has a derivative at every point, $f(x)$ is also continuous and derivable in $(P_{r-1},\ P_r)$

$\therefore$ By Mean Value Theorem, there exists a point ξ_r in $x_{r-1}<\xi_r<x_r$ such that

$$f(x_r)-f(x_{r-1})=(x_r-x_{r-1})f'(\xi_r)$$

or, $\quad \delta y_r = \delta x_r\, f'(\xi_r)$

$$\Rightarrow \quad \frac{\delta y_r}{\delta x_r} = f'(\xi_r).$$

$\therefore$ From (1) the length of the chord $P_{r-1}\, P_r$

$$= \sqrt{1+\{f'(\xi_r)\}^2}\ \delta x_r$$

So for any mode of division of (a, b), the limit of the sum of all the broken lines from P to Q as n tends to infinity exists and consequently the curve is rectifiable from P to Q.

Also the length of the curve from P to Q i.e., between $x=a$ and $x=b$ $(b>a)$ is

$$S = \operatorname*{Lt}_{n\to\infty} \sum_{r=1}^{n} \sqrt{1+\{f'(\xi_r)\}^2}\,\delta x_r$$

$$= \int_a^b \sqrt{1+\{f'(x)\}^2}\,dx$$

$$= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx. \qquad \cdots \qquad \cdots \qquad (2)$$

Cor. 1. If the curve be $x=f(y)$ and if $f(y)$ be continuous and derivable at every point, then the length of the area between $y=c$ and $y=d$ is

$$S=\int_{c}^{d}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy. \quad \cdots \quad \cdots \quad (3)$$

Cor. 2. For parametric equation of the curve

$$x=f(t),\ y=\phi(t)$$

$$\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}$$

$\therefore$ From (2)

$$S=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

where t_1 and t_2 are the values of t corresponding to the values of $x=a$ and b.

14.3. If integrated between proper limits the length of a curve can be obtained from the differential equation

$$dS=\sqrt{(dx)^2+(dy)^2}.$$

Let $P(x, y)$ be any point on the curve $y=f(x)$ at a curve distance S from some fixed point $A(a, b)$ on the curve. Let the curve be continuous and the tangent at P makes an angle ψ with the x-axis.

Then $\dfrac{dS}{dx}=\sec\psi=\sqrt{1+\tan^2\psi}=\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

So, by integration

$$S=\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx+C, \quad \cdots \quad \cdots \quad (1)$$

where C is the constant of integration

Let the indefinite integral $\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ be denoted by $F(x)$. Then (1) gives

$$S=F(x)+C. \quad \cdots \quad \cdots \quad (2)$$

If $P(x, y)$ coincides with A where $x=a$, then $S=0$

$\therefore$ From (2) $O=F(a)+C$

$\Rightarrow$ $C=-F(a)$.

So, from (2)

$$S=F(x)-F(a)$$

$$=\int_a^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Thus, if C and D are two point on the curve having abscissae $x=x_1$ and x_2 respectively $(x_2>x_1)$ and if S_1 and S_2 be their arc distances from $A(a, b)$ then

$$S_1=\int_a^{x_1} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

and $$S_2=\int_a^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

$$\therefore\quad S_2-S_1=\int_a^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx-\int_a^{x_1} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \dots \qquad \dots \qquad (3)$$

In parametric equation $x=f(t)$, $y=\phi(t)$, we can say S is function of t and so, we write

$$\frac{dS}{dt}=\frac{dS}{dx}\cdot\frac{dx}{dt}$$

$$=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot\frac{dx}{dt}$$

$$=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} \qquad \because\ \frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}.$$

$$\text{or,}\quad S=\int\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt+c$$

$$=F(t)+c\ (\text{say})\qquad\cdots\quad(4)$$

Suppose, the value of t is t_0 when P coincides with $A(a, b)$

$\therefore$ When $t=t_0$, $S=0$

So, from (4)

$$0=F(t_0)+c$$

$$\Rightarrow\quad c=-F(t_0)$$

$\therefore$ From (4)

$$S=F(t)-F(t_0)$$

$$=\int_{t_0}^{t}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

Thus, if S_1 and S_2 be the length of the areas from t_0 to t_1 and t_0 to t_2 respectively, then

$$S_1=\int_{t_0}^{t_1}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$\text{and}\quad S_2=\int_{t_0}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

$$\therefore\quad S_2-S_1=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.\qquad\cdots\quad(5)$$

(3) and (5) can be put in the differential from

$$dS=\sqrt{(dx)^2+(dy)^2}$$

which when integrated with respect to x between the limits x_1, and x_2, we get (3). Also when integrated with respect to t between the limits t_1 and t_2, we get (5).

Illustrations

Ex. 1. Find the perimeter of the circle $x^2+y^2=a^2$.

Differentiating the given equation w. r. to x

$$2x+2y\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Since, $\frac{1}{4}$ of the perimeter is covered with the limits $x=0$ and $x=a$

$$\therefore \quad S = 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 4\int_0^a \sqrt{1+\frac{x^2}{y^2}}\, dx = 4\int_0^a \frac{a}{y}\, dx$$

$$= 4a\int_0^a \frac{1}{\sqrt{a^2-x^2}}\, dx$$

$$= 4a\left[\sin^{-1}\frac{x}{a}\right]_0^a = 4a\frac{\pi}{2} = 2\pi a.$$

Otherwise.

Let the parametric equation be

$$x = a\cos t,\ y = a\sin t.$$

When t varies from 0 to $\frac{\pi}{2}$, $\frac{1}{4}$ the of the length of the curve is described.

$$\therefore \quad S = 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 4\int_0^{\pi/2} \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt$$

$$= 4a\int_0^{\pi/2} dt = 4a\frac{\pi}{2} = 2\pi a.$$

Ex. 2. Find the length of the arc of the parabola $y^2=4ax$ measured from the vertex to one end of the latus-rectum.

Here $2y\dfrac{dy}{dx}=4a$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{2a}{y}.$$

The co-ordinates of the upper extremity of the latus-rectum are $(a, 2a)$. So, the required length of the curve is covered within the limits $x=0$ and $x=a$

$$\therefore \quad S=\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_0^a \sqrt{1+\frac{4a^2}{y^2}}\,dx=\int_0^a \sqrt{\frac{4ax+4a^2}{4ax}}\,dx$$

$$=\int_0^a \sqrt{\frac{x+a}{x}}\,dx$$

$$=\int_0^a \frac{x+a}{\sqrt{x(x+a)}}\,dx$$

$$=\int_0^a \frac{\frac{1}{2}(2x+a)+\frac{1}{2}a}{\sqrt{x(x+a)}}\,dx$$

$$=\frac{1}{2}\int_0^a \frac{2x+a}{\sqrt{x(x+a)}}\,dx+\frac{1}{2}a\int \frac{dx}{\sqrt{x(x+a)}} \quad \ldots \quad (1)$$

[Put $x(x+a)=z$ in 1st integral then $(2x+a)dx=dz$]

$$\therefore \quad \int \frac{2x+a}{\sqrt{x(x+a)}}\, dx = \int \frac{dz}{\sqrt{z}} = \frac{z^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{z} = 2\sqrt{x(x+a)}.$$

[Put $x+a=z^2$ in 2nd integral, then $dx=2zdz$]

$$\therefore \quad \int \frac{dx}{\sqrt{x(x+a)}} = \int \frac{2z\, dz}{z\sqrt{z^2-a}} = 2\int \frac{dz}{\sqrt{z^2-a}}$$

$$= 2\log\left(z+\sqrt{z^2-a}\right)$$

$$= 2\log\left(\sqrt{x+a}+\sqrt{x}\right)$$

$\therefore$ From (1)

$$S = \left[\sqrt{x(x+a)} + a\log\left(\sqrt{x}+\sqrt{x+a}\right)\right]_a^a$$

$$= a\{\sqrt{2} + \log(1+\sqrt{2})\}.$$

Ex. 3. Rectify the ellipse $x=a\cos\theta$, $y=b\sin\theta$.

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (-a\sin\theta)^2 + (b\cos\theta)^2$$

$$= a^2\sin^2\theta + a^2(1-e^2)\cos^2\theta$$

$$= a^2\{1-\cos^2\theta + \cos^2\theta - e^2\cos^2\theta\}$$

$$= a^2(1-e^2\cos^2\theta)$$

Since, $\frac{1}{4}$th of the parameter is rectified within the limits $x=0$ and $x=\pi/2$, the total length of the parameter

$$= 4\int_0^{\pi/2} \left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}^{\frac{1}{2}} d\theta$$

$$= 4a\int_0^{\pi/2} \sqrt{(1-e^2\cos^2\theta)}\, d\theta.$$

$$= 4a\int_0^{\pi/2} \left\{1 - \frac{1}{2}e^2\cos^2\theta - \frac{1}{8}e^4\cos^4\theta - \frac{1}{16}e^6\cos^6\theta \cdots\right\} d\theta$$

$$= 4a\left\{\frac{\pi}{2} - \frac{1}{2}e^2\frac{1.\pi}{2\,2} - \frac{1}{8}e^4\frac{3.1}{4.2}\cdot\frac{\pi}{2} - \frac{1}{16}e^6\frac{5\,3.1}{6.4.2}\frac{\pi}{2}\cdots\cdots\right\}$$

$$=2a\pi\left\{1-\left(\frac{1}{2}\right)^2\frac{e^2}{1}-\left(\frac{1.3}{2.4}\right)^2\frac{e^4}{3}-\left(\frac{1.3.5}{2.4.6}\right)^2\frac{e^6}{5}-\cdots\right\}.$$

Ex. 4. If for a curve, $x\sin\theta+y\cos\theta=f'(\theta)$ and $x\cos\theta-y\sin\theta=f''(\theta)$, then show that

$S=f(\theta)+f''(\theta)+c$, where c is a constant.

(C.H. 1969, 67)

Multiplying 1st. and 2nd. equation by $\sin\theta$ and $\cos\theta$ respectively

$$x\sin^2\theta+y\sin\theta\cos\theta=f'(\theta)\sin\theta$$
$$x\cos^2\theta-y\sin\theta\cos\theta=f''(\theta)\cos\theta$$

$\therefore$ Adding $x=f'(\theta)\sin\theta+f''(\theta)\cos\theta \quad\cdots\quad (1)$

Similarly, multiplying 1st. and 2nd. equation by $\cos\theta$ and $\sin\theta$ respectively and subtracting

$$y=f'(\theta)\cos\theta-f''(\theta)\sin\theta \quad\cdots\quad (2)$$

$$\therefore\quad dx=\{f''(\theta)\sin\theta+f'(\theta)\cos\theta+f'''(\theta)\cos\theta-f''(\theta)\sin\theta\}\,d\theta$$
$$=\{f'(\theta)+f'''(\theta)\}\cos\theta\,d\theta.$$
$$dy=\{f''(\theta)\cos\theta-f'(\theta)\sin\theta-f'''(\theta)\sin\theta-f''(\theta)\cos\theta\}d\theta$$
$$=-\{f'(\theta)+f'''(\theta)\}\sin\theta\,d\theta.$$

Now $S=\int\{(dx)^2+(dy)^2\}^{\frac{1}{2}}$

$$=\int[\{f'(\theta)+f'''(\theta)\}^2\cos^2\theta+\{f'(\theta)+f'''(\theta)\}^2\sin^2\theta]^{\frac{1}{2}}d\theta.$$
$$=\int\{f'(\theta)+f'''(\theta)\}d\theta$$
$$=f(\theta)+f''(\theta)+c.$$

14.4. Length of the plane curve in polar co-ordinates :

(a) If S be the length of the curve from (r_1, θ_1) to (r_2, θ_2) of an equation $r=f(\theta)$, then (C.H. 1968)

$$S=\int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

Let $P(r, \theta)$ be a variable point on the curve $r=f(\theta)$ and let S be the length of the curve from some fixed point A

on the curve up to the variable point P. Also let ϕ be the angle made by the radius vector with the tangent at P. Then, we have

$$\tan\phi = r\frac{d\theta}{dr},\quad \cos\phi = \frac{dr}{dS},\quad \sin\phi = r\frac{d\theta}{dS}.$$

Now $\sin\phi = r\dfrac{d\theta}{dS}$

$$\Rightarrow \quad \frac{1}{r}\frac{dS}{d\theta} = \operatorname{cosec}\phi$$

$$= \sqrt{1+\cot^2\phi}$$

$$= \sqrt{1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2}$$

$$\Rightarrow \quad \frac{dS}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}.$$

So, integrating with respect to θ

$$S = \int \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta + c.$$

$$= F(\theta)+c \text{ (say)} \qquad \cdots \qquad \cdots \qquad (1)$$

Let $S=S_1$ when $\theta=\theta_1$

and $S=S_2$ when $\theta=\theta_2$

$\therefore$ From (1) $S_1 = F(\theta_1)+c$

$S_2 = F(\theta_2)+c$

$\therefore$ $S_2 - S_1 = F(\theta_2) - F(\theta_1)$

$$= \int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta. \qquad \cdots \qquad (2)$$

$$\Rightarrow \quad S = \int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

(b) If S be the length of the curve $r=f(\theta)$ from (r_1, θ_1) to (r_2, θ_2) then

$$S=\int_{r_1}\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr.$$

Here $\quad \dfrac{dr}{dS}=\cos\phi$

$$\Rightarrow \quad \frac{dS}{dr}=\sec\phi$$

$$=\sqrt{1+\tan^2\phi}=\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}$$

So, integrating with respect to r

$$S=\int\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr+c$$

$$=F(r)+c \text{ (say)} \qquad \cdots \quad (3)$$

Let $\quad S=S_1 \quad$ when $r=r_1$

and $\quad S=S_2 \quad$ when $r=r_2$

So, from (1)

$$S_1=F(r_1)+c$$

and $\quad S_2=F(r_2)+c$

$\therefore \quad S_2-S_1=F(r_2)-F(r_1)$

$$=\int_{r_1}^{r_2}\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr \qquad \cdots \quad (4)$$

$$\Rightarrow \quad S=\int_{r_1}^{r_2}\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr.$$

Cor. Equation (2) and (4) can be put in a convenient differential form

$$dS=\sqrt{dr^2+r^2d\theta^2}$$

which when integrated with respect to θ from θ_1 to θ_2, we get (2) and when integrated with respect to r from r_1 to r_2, we get (4).

Illustrations :

Ex. 1. Find the length of the arc of the angular spiral $r=ae^{\theta\cot\alpha}$ between $r=r_1$ and $r=r_2$.

Differentiating the given equation with respect to θ

$$\frac{dr}{d\theta}=a\cot\alpha\,.\,e^{\theta\cot\alpha}=r\cot\alpha.$$

$$\Rightarrow\quad \frac{d\theta}{dr}=\frac{1}{r}\tan\alpha$$

or, $$r^2\left(\frac{d\theta}{dr}\right)^2=\tan^2\alpha$$

$$\therefore\quad S=\int_{r_1}^{r_2}\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr$$

$$=\int_{r_1}^{r_2}\sqrt{1+\tan^2\alpha}\,dr.=\sec\alpha\int_{r_1}^{r_2}dr=(r_2-r_1)\sec\alpha.$$

Ex. 2. Find the length of the arc of the parabola

$$r=\frac{2}{1+\cos\theta}$$

measured from $\theta=0$ to $\theta=\pi/2$.

The given equation is $r=\sec^2\frac{\theta}{2}$.

$\therefore$ Differentiating with respect to θ

$$\frac{dr}{d\theta}=\sec^2\frac{\theta}{2}\,.\,\tan\frac{\theta}{2}=r\tan\frac{\theta}{2}$$

$$\therefore\quad S=\int_0^{\pi/2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$=\int_0^{\pi/2}\sqrt{r^2+r^2\tan^2\frac{\theta}{2}}\,d\theta.$$

$$=\int_0^{\pi/2}\sqrt{1+\tan^2\frac{\theta}{2}}\,.\,r\,d\theta$$

$$=\int_0^{\pi/2}\sqrt{1+\tan^2\frac{\theta}{2}}\,\sec^2\frac{\theta}{2}\,d\theta.$$

$$=2\int_0^1 \sqrt{1+z^2}\,dz. \qquad \left[\text{Putting } \tan\frac{\theta}{2}=z.\right]$$

$$=2\left[\frac{z\sqrt{z^2+1}}{2}+\tfrac{1}{2}\log\left(z+\sqrt{z^2+1}\right)\right]_0^1$$

$$=2\left[\frac{\sqrt{2}}{2}+\tfrac{1}{2}\log(1+\sqrt{2})\right]=\sqrt{2}+\log(1+\sqrt{2}).$$

14.5. Length of a curve from a given point up to a variable point P on the curve in terms of the angle made by the tangent at P with a fixed line.

If ψ be the angle made by the tangent at a variable point P on the curve with the x-axis, and S be the length of the curve from some fixed point A on the curve up to the variable point P on the curve, then any relation that exists between S and ψ is called the intrinsic equation to the curve.

(I) How to derive Intrinsic equation from the cartesian equation.

Let $P(x, y)$ be any point on the curve $y=f(x)$ and let the tangent at P makes an angle ψ with the x-axis.

$$\text{Then,} \quad \tan\psi=\frac{dy}{dx}=f'(x) \qquad \cdots \qquad \cdots \qquad (1)$$

From this an equation connecting x and ψ is obtained.

Again, if S denotes the length of the curve measured from some fixed point $A(a, b)$ on the curve up to the point P, then

$$S=\int_a^x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_a^x \sqrt{1+\{f'(x)\}^2}\,dx$$

$$=F(x) \text{ say.} \quad \cdots \quad (2)$$

From this an equation connecting S and x is obtained.

Now solving (1) and (2) and eliminating x, we shall get an equation connecting S and ψ.

Hence, x eliminant of (1) and (2) is the required intrinsic equation of the curve.

If $x=f(t)$, $y=g(t)$ be the parametric equation of the curve, then

$$\tan\psi=\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=g'(t)/f'(t) \quad \cdots \quad (3)$$

From this, an equation connecting ψ and t is obtained.

$$\text{Again } S=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$=\int_{t_1}^{t_2}\sqrt{\{f'(t)\}^2+\{g'(t)\}^2}\,dt \qquad \text{[Where } t_1 \text{ is the value of } t \text{ at } A.]$$

$$=F(t) \text{ (say)} \quad \cdots \quad \cdots \quad (4)$$

From this, an equation connecting S and t is obtained.

Now, eliminating t from (3) and (4) an equation connecting S and ψ is obtained which is, therefore, the intrinsic equation of the curve.

(II) How to derive Intrinsic equation from the polar equation.

Let $P(r, \theta)$ be a variable point on the curve $r=f(\theta)$. Let the tangent at P make an angle ψ with the axis and ϕ be the angle between the radius vector at P and the tangent at P.

Then $\psi=\theta+\phi \quad \cdots \quad \cdots \quad (1)$

$$\tan \phi = r\frac{d\theta}{dr} = r\Big/\frac{dr}{d\theta} = \frac{f(\theta)}{f'(\theta)} \quad \cdots \quad \cdots \quad (2)$$

And $$S = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_{\alpha}^{\beta} \sqrt{\{f(\theta)\}^2 + \{f'(\theta)\}^2}\, d\theta$$

$$= F(\theta) \text{ (say)} \quad \cdots \quad \cdots \quad (3)$$

Now, eliminating θ and ϕ from (1), (2) and (3), a relation connecting S and ψ is obtained which is the required intrinsic equation of the curve.

(III) How to derive Intrinsic equation from pedal equation.

Let $p = f(r)$ be the pedal equation of a curve.

If S be the length of the curve between the radius vectors a and r, then

$$S = \int_{a}^{r} \frac{r\, dr}{\sqrt{r^2 - p^2}}$$

$$= \int_{a}^{r} \frac{r\, dr}{\sqrt{r^2 - \{f(r)\}^2}} = F(r) \text{ say} \quad \cdots \quad \cdots \quad (1)$$

From this, an equation connecting S and r is obtained.

Again, from $\rho = \dfrac{dS}{d\psi} = r\dfrac{dr}{dp}$

we get, $$\frac{dS}{d\psi} = r\Big/\frac{dp}{dr} = r\big/f'(r) \quad \cdots \quad \cdots \quad (2)$$

From this, an equation of $\dfrac{dS}{d\psi}$ and r is obtained.

So, eliminating r from (1) and (2), an equation connecting $\dfrac{dS}{d\psi}$ and S is obtained.

Let this relation be

$$\frac{dS}{d\psi}=g(S)$$

$$\Rightarrow \quad d\psi=\frac{dS}{g(S)}$$

Integrating $\psi=\int_{S_1}^{S_2}\frac{dS}{g(S)}+c$

$$=G(S_2)-G(S_1)$$

which gives the required intrinsic equation of the curve.

Illustrations :

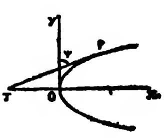
Ex. 1. Find the intrinsic equation of the parabola $y^2=4ax$ taking origin as the fixed point.

Here, y axis is the tangent at the vertex. Let $P(x, y)$ be any point on the parabola at an arc distance S from the vertex. Let the tangent at P make an angle ψ with the y axis.

Then, $\tan\psi=\frac{dx}{dy}=\frac{y}{2a}$ ... ... (1)

Also $S=\int_0^y \sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\,dy$

$$=\int_0^y \sqrt{\left\{1+\frac{y^2}{4a^2}\right\}}\,dy$$



$$=\frac{1}{2a}\int_0^y \sqrt{(4a^2+y^2)}\,dy$$

$$=\frac{1}{2a}\left[\frac{y\sqrt{4a^2+y^2}}{2}+\frac{4a^2}{2}\log\left(y+\sqrt{4a^2+y^2}\right)\right]_0^y$$

$$=\frac{1}{2a}\left[\frac{y\sqrt{4a^2+y^2}}{2}+2a^2 \log \frac{y+\sqrt{4a^2+y^2}}{2a}\right] \quad \cdots (2)$$

Eliminating y from (1) and (2) we get

$$S=\frac{1}{2a}\left[\frac{2a\tan\psi\sqrt{4a^2+4a^2\tan^2\psi}}{2}\right.$$

$$\left.+2a^2 \log \frac{2a\tan\psi+\sqrt{4a^2+4a^2\tan^2\psi}}{2a}\right]$$

$$=a[\tan\psi\sec\psi+\log(\tan\psi+\sec\psi)].$$

which is the required intrinsic equation of the parabola.

Ex. 2. Find the intrinsic equation of the semicubical parabola $ay^2=x^3$ taking its cusp as the fixed point.

Here, origin is the cusp and the tangent at the cusp is the x axis.

Let $P(x, y)$ be any point on the curve at an arc distance S from the cusp and let the tangent at P make an angle ψ with the x axis.

Let the parametric equation of the curve be $x=at^2$, $y=at^3$.

Then $\quad dx=2at\ dt.$

$\qquad\quad dy=3at^2\ dt.$

Now $\qquad \tan\psi=\frac{dy}{dx}=\frac{3}{2}t$

$$\Rightarrow \quad t=\frac{2}{3}\tan\psi \qquad \cdots \quad (1)$$

Again, since $t=0$ at the cusp

$$S=\int_0^t \{(dx)^2+(dy)^2\}^{\frac{1}{2}}$$

$$=a\int_0^t (4+9t^2)^{\frac{1}{2}}\ .\ t\ dt. \qquad (2)$$

Now, eliminating t from (1) and (2), we get

$$S=a\int^{\psi}(4+4\tan^2\psi)^{\frac{1}{2}}.\frac{2}{3}\tan\psi.\frac{2}{3}\sec^2\psi\ d\psi$$

$$=\frac{8a}{9}\int_0^{\psi}\sec^3\psi\tan\psi\,d\psi=\frac{8a}{27}\Big[\sec^3\psi\Big]_0^{\psi}$$

$$=\frac{8a}{27}(\sec^3\psi-1)$$

or, $27S=8a(\sec^3\psi-1)$ is the required intrinsic equation to the curve.

Exercise 14

1. Show that the arc of the semi-cubical parabola $9y^2=4x^3$ from the cusp to any point (x, y) is $\frac{2}{3}\left\{(1+x)^{\frac{3}{2}}-1\right\}$.

2. Show that the length of the arc of the curve $y=2x-x^2$ as x varies from 0 to 2 is

$$\sqrt{5}+\tfrac{1}{2}\log(2+\sqrt{5}).$$

3. Show that the entire length of the astroid

$x=a\cos^3 t,\ y=\sin^3 t$ is $6a$.

Deduce that the length S of the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ measured from $(0, a)$ to any point (x, y) is $S=\frac{3}{2}\sqrt[3]{ax^2}$.

4. Prove that the length of the arc of the parabola $\frac{l}{r}=1+\cos\theta$ cut off by its latus-rectum is $l\{\sqrt{2}+\log(1+\sqrt{2})\}$ (C. H. 1973)

5. Find the length of the curve $r=a\cos^3\frac{\theta}{3}$. Ans. $\frac{3}{2}\pi a$.

6. Show that the perimeters of the two ellipses

$\frac{x^2}{9}+\frac{y^2}{7}=1$ and $\frac{x^2}{36}+\frac{y^2}{28}=1$ are in the ratio 1 : 2. (C. H. 1925)

7. Prove that the length of the curve $x=e^t\sin t,\ y=e^t\cos t$ from $t=0$ and $t=\frac{\pi}{2}$ is $\sqrt{2}(e^{\pi/2}-1)$.

8. Prove that the perimeter L of an ellipse with semi-axes a and b is given by (C. H. 1970)

$$=4\int_0^{\pi/4}\left\{\sqrt{(a^2\cos^2 t+b^2\sin^2 t)}+\sqrt{(a^2\sin^2 t+b^2\cos^2 t)}\right\}dt$$

Deduce that $\pi(a+b)<L<\pi\sqrt{2}\sqrt{(a^2+b^2)}$.

9. Show that the length of the curve

$x=a(\sin\theta+\frac{1}{3}\sin 3\theta)$, $y=a(\cos\theta-\frac{1}{3}\cos 3\theta)$ is $8a$. (B. H. 1970)

10. If S denotes the length of the arc of the curve $x=a(\theta+\sin\theta\cos\theta)$, $y=a(1+\sin\theta)^2$ measured from the point $\theta=-\pi/2$ to any point θ, prove that S^4 varies as y^3. (B. H. 1971)

11. Show that the arc of the upper half of the cardioide $r=a(1-\cos\theta)$ is bisected at $\theta=\frac{2}{3}\pi$. Also show that the perimeter of the curve is $8a$. (C.H. 1971)

12. If S be the length of the curve $x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$ from $\theta=0$ to α, then, show that S varies as α^2.

13. Show that the length of the arc of the parabola $\frac{1}{r}=1+\cos\alpha$ cut off by its latus-rectum is $\{\sqrt{2}+\log(1+\sqrt{2})\}$. (N. B. 1969)

14. If S denotes the total length of the evolute of the ellipse $x^2/a^2+y^2/b^2=1$ show that $ab.S=4(a^3-b^3)$.

15. Show the length of the parabola $p^2=ar$ from $r=a$ to $r=2a$ is

$$a\{\sqrt{2}+\log(1+\sqrt{2})\}.$$

16. Let S_1 be the length of the arc of the hyperbola $xy=c^2$ between the points $x=\alpha$ and $x=\beta$ and S_2 be the length of the arc of the curve $p^2(c^4+r^4)=c^4r^2$ between the point $r=\alpha$ and $r=\beta$. Show that $S_1=S_2$.

17. Show that the area between the curve

$$y=c\cosh\frac{x}{c},$$

the x axis and the ordinates at two points on the curve is equal to c times the length of the arc terminated by those points.

18. Find the length of the curve defined by

$x=a\cos t-\frac{1}{3}(a-b)\cos^3 t$, $y=b\sin t+\frac{1}{3}(a-b)\sin^3 t$

measured from the point $t=0$ to the point t. (C. H. 1962)

Ans. $\frac{1}{2}\{(a+b)t-\frac{1}{4}(a-b)\sin 2t\}$

CHAPTER XV

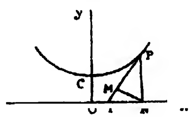
ON SOME WELL KNOWN CURVES AND THEIR QUADRATURE AND RECTIFICATION

15 1. Catenary or, Chainette

The catenary is the curve formed by a uniform heavy string when suspended under the action of gravity from two horizontal points. Its equation is

$$y=c\cosh\frac{x}{c}=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)$$

The directrix OX is the x axis and the vertex is at $C(0, c)$, so that $OC=c$. The characteristic properties of a catenary are (1) The length of the perpendicular from the foot of any ordinate on the tangent at that point is of constant length. (2) The length of the normal at any point is equal to the radius of curvature of that point.



Ex 1 Find the area included between the catenary

$$\mathbf{y=c\cosh\frac{x}{c}}$$

the x axis and the ordinates at two points on the curve.

Let the abscissa of any two points on the curve be x_1 and x_2 $(x_2>x_1)$

$$\text{Here,}\quad A\ (\text{area})=\int^{x_2} y\,dx$$

$$=\int_{x_1}^{x_2} c\cosh\frac{x}{c}dx$$

$$=\left[\sinh\frac{x}{c}\right]_{x_1}^{x_2}$$

$$=\sinh\left(\frac{x_2}{c}\right)-\sinh\left(\frac{x_1}{c}\right).$$

Ex. 2. Find the length of the catenary

$$y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right.$$

from the vertex to any point (x_1, y_1) on the curve.

(C. H. 1971)

Since, the vertex is at $(0, c)$, the required length of the catenary is enclosed within the limits $x=0$ and $x=x_1$

$$\therefore \quad S=\int_0^{x_1}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

$$=\int_0^{x_1}\sqrt{1+\sinh^2\frac{x}{c}}\,dx \qquad \left[\because \; y=c\cosh\frac{x}{c},\; \frac{dy}{dx}=\sinh\frac{x}{c}\right]$$

$$=\int_0^{x_1}\cosh\frac{x}{c}\,dx$$

$$=c\sinh\frac{x_1}{c}.$$

Ex. 3. Find the intrinsic equation of the catenary

$$y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)$$

taking the vertex as the fixed point.

$$y=c\cosh\frac{x}{c}$$

$$\therefore \quad \tan\psi=\frac{dy}{dx}=\sinh\frac{x}{c} \qquad (1)$$

Also, the length of the catenary from the vertex to any point $P(x, y)$ on the catenary is

$$S=\int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^x \sqrt{1+\sinh^2 \frac{x}{c}}\,dx$$

$$=\int_0^x \cosh \frac{x}{c}dx=c \sinh \frac{x}{c} \qquad \cdots \quad (2)$$

Eliminating, $\sinh \frac{x}{c}$ from (1) and (2)

$$S=c \tan \psi$$

which is, therefore, the required intrinsic equation to the catenary.

15.2. Astroid.

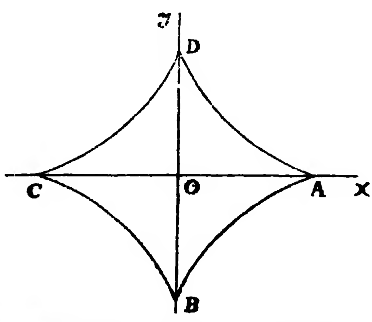
The envelope of the family of straight lines $\frac{x}{a}+\frac{y}{b}=1$ where the perimeters a and b are connected by the relation $a^2+b^2=$constant, is called an Astroid. So the characteristic property of an Astroid is that the tangent at any point of the curve intercepted between the two axis is of constant length.

The equation of the astroid is

$$x^{2/3}+y^{2/3}=a^{2/3}$$

and its parametric equation is

$$x=a \cos^3 \theta, y=a \sin^3 \theta.$$



It has four cusps all lying on the axes at A, B, C and D in such away that $OA=OB=OC=OD=a$. So, the astroid may be enclosed within a circle with centre at O and radius equal to OA so that all the cusps lying on the circumference.

Illustrations :

Ex. 1. Find the area of the astroid

$$x^{2/3}+y^{2/3}=a^{2/3}.$$

Let the parametric equation of the astroid be

$$x=a\cos^3\theta,\quad y=a\sin^3\theta.$$

The co-ordinates of the four cusps are

$$(a, 0), (0, a), (-a, 0), (0, -a)$$

So, from symmetry the total area = 4 times the area in the first quadrant.

$$\therefore\quad A=4\int_0^a y\,dx$$

$$=4\int_{\pi/2}^{0} a\sin^3\theta(-3a\cos^2\theta\sin\theta)\,d\theta$$

$$=12a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta.$$

$$=12a^2\int_0^{\pi/2}\sin^4\theta(1-\sin^2\theta)d\theta$$

$$=12a^2\int_0^{\pi/2}(\sin^4\theta-\sin^6\theta)d\theta$$

$$=12a^2\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$=12a^2\cdot\frac{3}{16}\pi\left(1-\frac{5}{6}\right)$$

$$=\frac{3}{8}\pi a^2.$$

Ex. 2. Find the perimeter of the astroid

$$x^{2/3}+y^{2/3}=a^{2/3}$$

Let $x=a\cos^3\theta$, $y=a\sin^3\theta$ be the parametric equation of the astroid.

Then $dx=-3a\cos^2\theta\sin\theta\,d\theta$
$dy=3a\sin^2\theta\cos\theta\,d\theta$
$S=4\times$ length of the curve in first quadrant.

$$=4\int_0^{\pi/2}\{(dx)^2+(dy)^2\}^{\frac{1}{2}}$$

$$=4\int_0^{\pi/2}\{9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta\}^{\frac{1}{2}}d\theta$$

$$=12a\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta$$

$$=12a\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}=6a.$$

Ex. 3. Find the intrinsic equation of the astroid
$$\mathbf{x^{2/3}+y^{2/3}=a^{2/3}}$$
arc being measured from one of the cusps.

Let the parametric equation of the astroid be
$$x=a\cos^3\theta,\quad y=a\sin^3\theta.$$

$\therefore\quad dx=-3a\cos^2\theta\sin\theta\,d\theta\ ;\quad dy=3a\sin^2\theta\cos\theta\,d\theta$

So, $\tan\psi=\dfrac{dy}{dx}=-\tan\theta$

$\therefore\quad \theta=-\psi \quad \cdots \quad \cdots \quad (1)$

$$S=\int_0^{\theta}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$=\int_0^{\theta}\sqrt{9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta}\,d\theta$$

$$=3a\int^{\theta}\sin\theta\cos\theta\,d\theta$$

$$=3a\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\theta}$$

$$=\tfrac{3}{2}a\sin^2\theta \qquad \cdots \qquad \cdots \qquad (2)$$

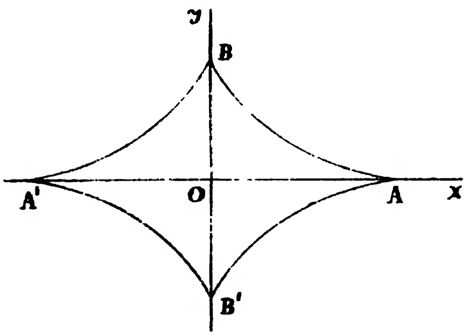
Eliminating θ from (1) and (2), we get

$$S=\tfrac{3}{2}a\sin^2(-\psi)$$
$$=\tfrac{3}{2}a\sin^2\psi$$

which is the required intrinsic equation of the curve.

15.3. Hypo-cycloid.

It is a general form of astroid containing four cusps all lying on an ellipse at the extremities of the major and minor axes.



Its equation is

$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$$

and parametric equation $x=a\cos^3\theta$, $y=b\sin^3\theta$.

Here $AA'=2a$, and $BB'=2b$

So $\quad OA=a, \quad OB=b$

Illustrations :

Ex. 1. Find the area enclosed by the four cusped hypocycloid $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$.

Let the parametric equation be

$$x=a\cos^3\theta, \quad y=b\sin^3\theta.$$

The coordinates of the four cusps are

$$(a, 0), (0, b), (-a, 0), (0, -b).$$

So, in the positive quadrant as x varies from 0 to a, θ varies from $\pi/2$ to 0.

$$\therefore \quad A=4\int^{a} y\,dx$$

$$=4\int_{\pi/2}^{0} b\sin^3\theta\,(-3a\cos^2\theta\,\sin\theta)\,d\theta.$$

$$=12ab\int^{\pi/2}\sin^4\theta(1-\sin^2\theta)\,d\theta$$

$$=12ab\int_0^{\pi/2}(\sin^4\theta-\sin^6\theta)\,d\theta.$$

$$=12ab\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$=12ab\,.\,\frac{3\pi}{16}\left(1-\frac{5}{6}\right)$$

$$=\frac{3}{8}\pi ab.$$

Ex. 2. Find the perimeter of the hypo cycloid

$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1.$$

Let the parametric equation be

$$x=a\cos^3\theta,\quad y=b\sin^3\theta.$$

Then $\quad dx=-3a\cos^2\theta\sin\theta\,d\theta,\ dy=3b\sin^2\theta\cos\theta\,d\theta.$

In the first quadrant as x varies from 0 to a, θ varies from $\pi/2$ to 0.

$\therefore \quad S=4\times$ length of the curve in the 1st quadrant

$$=\int^{a}\{(dx)^2+(dy)^2\}^{\frac{1}{2}}$$

$$=4\int_{\pi/2}^{0}\{9a^2\cos^4\theta\sin^2\theta+9b^2\sin^4\theta\cos^2\theta\}^{\frac{1}{2}}\,d\theta$$

$$=12\int_{\pi/2}^{0}\sin\theta\cos\theta\,(a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{1}{2}}d\theta.$$

[Put $z^2=a^2\cos^2\theta+b^2\sin^2\theta$

$\therefore\quad z\,dz=(-a^2\cos\theta\sin\theta+b^2\sin\theta\cos\theta)\,d\theta$

$\qquad\qquad = -(a^2-b^2)\sin\theta\cos\theta\,d\theta$]

$$\therefore\quad S=\frac{12}{a^2-b^2}\int_b^a z^2\,dz=\frac{12}{a^2-b^2}\left[\frac{z^3}{3}\right]_b^a$$

$$=\frac{4}{a^2-b^2}(a^3-b^3)=4.\,\frac{a^2+ab+b^2}{a+b}.$$

Ex. 3. Find the intrinsic equation of the hypo-cycloid

$$\left(\frac{\mathbf{x}}{\mathbf{a}}\right)^{2/3}+\left(\frac{\mathbf{y}}{\mathbf{b}}\right)^{2/3}=1.$$

Let the parametric equation be

$$x=a\cos^3\theta,\quad y=b\sin^3\theta$$

$\therefore\quad dx=-3a\cos^2\theta\sin\theta\,d\theta,\ dy=3b\sin^2\theta\cos\theta\,d\theta$

So, $\quad \tan\psi=\dfrac{dy}{dx}=-\dfrac{b}{a}\tan\theta$

$\Rightarrow\quad \tan\quad =-\dfrac{a}{b}\tan\psi \qquad \cdots \qquad \cdots \qquad (1)$

Now $\quad S=\displaystyle\int_0^\theta\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$

$$=\int_0^\theta\sqrt{9a^2\cos^4\theta\sin^2\theta+9b^2\sin^4\theta\cos^2\theta}\,d\theta.$$

$$=3\int\sin\theta\cos\theta\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}$$

[Put $z^2=a^2\cos^2\theta+b^2\sin^2\theta$

$\therefore \quad zdz = (-a^2 \cos\theta \sin\theta + b^2 \sin\theta \cos\theta)\, d\theta$
$= (b^2 - a^2) \sin\theta \cos\theta\, d\theta]$

$$= \frac{3}{b^2 - a^2} \int_a^{\alpha} z^2\, dz$$

$\because$ when $\theta = 0$, $z = a$
$\theta = \theta$, $z = \pm \sqrt{a^2 \cos^2\theta + b^2 \cos^2\theta}$
$= \alpha$ (say).]

$$= \frac{3}{b^2 - a^2} \frac{\alpha^3 - a^3}{3} \qquad \ldots \qquad \ldots \qquad (2)$$

where $\alpha^3 = \pm (a^2 \cos^2\theta + b^2 \sin^2\theta)^{3/2}$

$$= \pm \left\{ \frac{a^2}{1 + \tan^2\theta} + \frac{b^2}{1 + \cot^2\theta} \right\}^{3/2}$$

$$= \pm \left\{ \frac{a^2}{1 + \frac{a^2}{b^2} \tan^2\psi} + \frac{b^2}{1 + \frac{b^2}{a^2} \cot^2\psi} \right\}^{3/2}$$

$$= \pm a^3 b^3 \left\{ \frac{1}{b^2 + a^2 \tan^2\psi} + \frac{1}{a^2 + b^2 \cot^2\psi} \right\}^{3/2}.$$

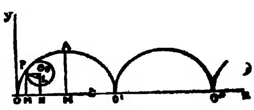
Hence, from (2) the required intrinsic equation is

$$S = \frac{1}{b^2 - a^2} \left[\pm a^3 b^3 \left\{ \frac{1}{b^2 + a^2 \tan^2\psi} + \frac{1}{a^2 + b^2 \cot^2\psi} \right\} - a^3 \right].$$

15.4. Cycloid.

The curve traced out on the plane of the paper by a point on the circumference of a circle which rolls, without sliding, on a straight line is called a cycloid.

Let $x^2 + y^2 = a^2$ be the equation of the generating circle with a point P on it. When the circle rolls along the x-axis through a distance ON, let the point trace out an arc OP so that $ON = \text{arc } PN$ of the circle $= a\theta$. Let (x, y) be the co-ordinates of P at this moment with respect to OX and OY as axes.



(fig. i)

Then $\quad x = OM = ON - MN = ON - PL = a\theta - a \sin\theta$
$= a(\theta - \sin\theta)$.

$y = PM = CN - CL = a - a\cos\theta = a(1 - \cos\theta)$.

Hence, the parametric equation of the cycloid with the starting point as its origin, the straight line on which it rolls as the axis of x are

$$x=a(\theta-\sin\theta),\quad y=a(1-\cos\theta).$$

Let A be the highest point of one arc of the cycloid. Then A is called its vertex. At the vertex y is maximum i.e., $a(1-\cos\theta)$ is maximum.

$\Rightarrow$ $\cos\theta=-1$ i.e., $\theta=\pi$.

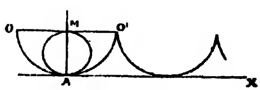
So that $OM=a(\pi-\sin\pi)=a\pi$

$AM=a(1-\cos\pi)=2a$

$\therefore$ vertex is at $(a\pi, 2a)$.

The co-ordinates of the point O' where the first arc of the cycloid meet the x-axis are $(0, 2\pi)$.

If the generating circle rolls below the line called the x-axis then an inverted cycloid is traced out on the plane of the paper.



(fig. ii)

If the vertex A be chosen as the origin, and the tangent at A be the axis of x, the equation of this cycloid be similarly proved to be

$$x=a(\theta+\sin\theta),\quad y=a(1-\cos\theta).$$

Hence, $\theta=0$ at A, $\theta=\pi$ at O and $\theta=-\pi$ at O'.

Illustrations:

Ex. 1. Find the area of one arc of the cycloid

$$x=a(\theta+\sin\theta),\quad y=a(1-\cos\theta).$$

As y varies from 0 to $2a$, half of the total area is described.

$$\therefore \text{ Half the area}=\int_0^{2a} x\,dy$$

$$=\int_0^{\pi} a(\theta+\sin\theta)a\sin\theta\,d\theta$$

$$=a^2\int_0^{\pi} \theta \sin\theta + a^2\int_0^{\pi} \sin^2\theta \, d\theta$$

$$=a^2\left[-\theta\cos\theta+\sin\theta+\frac{1}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right)\right]_0^{\pi}=\frac{3}{2}\pi a^2.$$

$\therefore$ Total area $=3\pi a^2$.

Ex. 2. Rectify the curve

$$x=a(\theta+\sin\theta), \quad y=a(1-\cos\theta).$$

Here, $\dfrac{dx}{d\theta}=a(1+\cos\theta)$, $\dfrac{dy}{d\theta}=a\sin\theta$.

$$\therefore \quad \sqrt{\left\{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right\}}=\sqrt{2a^2(1+\cos\theta)}=2a\cos\frac{\theta}{2}.$$

As θ varies from 0 to π, half of the arc is described.

$$\therefore \quad \text{Required length}=2\int_0^{\pi} 2a\cos\frac{\theta}{2}\, d\theta$$

$$=8a\left[\sin\frac{\theta}{2}\right]_0^{\pi}=8a.$$

Ex. 3. Obtain the intrinsic equation of the cycloid

$$\mathbf{x=a(\theta+\sin\theta), \quad y=a(1-\cos\theta)}$$

taking origin as the fixed point.

Let $P(\theta)$ be any variable point on the cycloid.

Now $\dfrac{dx}{d\theta}=a(1+\cos\theta)$, $\dfrac{dy}{d\theta}=a\sin\theta$

$$\therefore \quad \tan\psi=\frac{dy}{dx}=\frac{\sin\theta}{1+\cos\theta}=\tan\frac{\theta}{2}$$

$$\Rightarrow \quad \psi=\frac{\theta}{2} \quad \cdots \quad \cdots \quad (1)$$

Also $\quad S=\displaystyle\int_0^{\theta}\sqrt{\left\{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right\}}d\theta$

$$=\int_0^\theta \sqrt{\{a^2(1+\cos\theta)^2+a^2\sin^2\theta\}}\,d\theta$$

$$=a\int_0^\theta \sqrt{1+\cos^2\theta+2\cos\theta+\sin^2\theta}\,d\theta$$

$$=a\int_0^\theta \sqrt{2(1+\cos\theta)}\,d\theta$$

$$=2a\int_0^\theta \cos\frac{\theta}{2}\,d\theta=4a\sin\frac{\theta}{2} \quad \cdots \quad (2)$$

Eliminating θ from (1) and (2) the required intrinsic equation is $S=4a\sin\psi$.

Ex. 4. In the cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ if arc is measured from the vertex where $\theta=0$, show that $\rho^2+S^2=16a^2$. (B. H. 1970, C. H. 1972)

From Ex. 3 above $S=4a\sin\psi$

$$\therefore \quad \rho=\frac{dS}{d\psi}=4a\cos\psi$$

$$\therefore \quad \rho^2+S^2=16a^2\cos^2\psi+16a^2\sin^2\psi=16a^2.$$

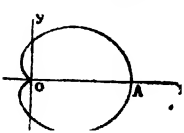
15.5. Cardioide.

The curve traced out by the pedal of the circle $r=2a\cos\theta$ is called a cardioide. The shape of the curve is like human heart and hence the name.

The equation of the pedal of $r=2a\cos\theta$

i.e., the equation of the cardioide (fig. I) is

$$r=a(1+\cos\theta)$$



(fig. I)

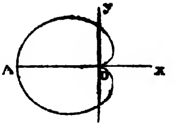
The curve has a cusp at O where there is a common tangent taken as the initial line i.e., x-axis. The curve is symmetrical about this initial line. For A, $\theta=0$. and for O, $\theta=\pi$.

If the above cardioide be turned through an angle 180°, the position of O and A are interchanged.

The equation of this cardioide (fig. II) is $r=a(1-\cos\theta)$.

for O, $\theta=0$

and for A, $\theta=\pi$



(fig. ii)

Illustrations :

Ex. 1. Find the area of the cardioide (C. H. 1970)

$$\mathbf{r=a(1+\cos\theta)}.$$

The curve is symmetrical about the initial line. Also as θ varies from 0 to π half the area is described.

$$\therefore\quad \text{Required area}=2.\tfrac{1}{2}\int_0^{\pi} r^2 d\theta$$

$$=\int_0^{\pi} a^2(1+\cos\theta)^2\, d\theta$$

$$=a^2\int_0^{\pi} 4\cos^4\frac{\theta}{2}\,d\theta \qquad \left[\text{Put } \frac{\theta}{2}=z \quad \therefore\ d\theta=2dz\right]$$

$$=8a^2\int_0^{\pi/2}\cos^4 z\, dz.$$

$$=8a^2\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{3}{2}\pi a^2.$$

Ex. 2. Find the perimeter of the cardioide (C. H. 1970)

$$\mathbf{r=a(1+\cos\theta)}.$$

Here $\dfrac{dr}{d\theta}=-a\sin\theta.$

$$\therefore \quad \sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}=\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}$$

$$=a\sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta}$$

$$=a\sqrt{2(1+\cos\theta)}$$

$$=2a\cos\frac{\theta}{2}.$$

The curve is symmetrical with respect to the initial line. Also as θ varies from 0 to π half the area is described.

$\therefore$ The required perimeter

$$=2\int_0^{\pi} 2a\cos\frac{\theta}{2}d\theta. \qquad \left[\text{Put } \frac{\theta}{2}=z\right.$$

$$\left.\therefore \quad d\theta=2dz\ \right]$$

$$=8a\int_0^{\pi/2}\cos z\,dz=8a.\Big[\sin z\Big]_0^{\pi/2}=8a.$$

Ex. 3. Obtain the intrinsic equation of the cardioide

(C. H. 1968)

$$r=a(1-\cos\theta)$$

taking pole as the fixed point.

$$\tan\phi=r\frac{d\theta}{dr}$$

$$=a(1-\cos\theta)\frac{1}{a\sin\theta}=\tan\frac{\theta}{2}$$

$$\Rightarrow \quad \phi=\frac{\theta}{2}.$$

But $\quad \psi=\theta+\phi$

$$=\theta+\frac{\theta}{2}=\frac{3}{2}\theta.$$

$$\therefore \quad \frac{\theta}{2}=\frac{\psi}{3} \qquad \dots \qquad \dots \qquad (1)$$

Again $\quad S=\int_0^{\theta}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta$

$$=2a\int \sin\frac{\theta}{2}\,d\theta$$

$$=-4a\left[\cos\frac{\theta}{2}\right]_0$$

$$=4a\left(1-\cos\frac{\theta}{2}\right) \qquad \cdots \qquad \cdots \qquad (2)$$

Eliminating $\frac{\theta}{2}$ from (1) and (2), the required intrinsic equation is

$$S=4a(1-\cos\psi/3).$$

15.6. Folium of Descartes.

The equation of the Folium of Descartes containing a loop in the first quadrant is

$$x^3+y^3=3axy.$$

It is symmetrical about the line $y=x$ and meets it at $(\frac{3}{2}a, \frac{3}{2}a)$. Origin is a node with $x=0$, $y=0$ as tangents at the node.

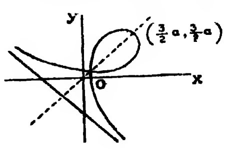
The curve has its only asymptote

$$x+y+a=0.$$

The equation in polar co-ordinates by writing $x=r\cos\theta$, $y=r\sin\theta$ is

$$r=\frac{3a\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}$$

As θ increases from 0 to $\pi/2$, the curve describes a loop in the first quadrant. As θ increases from $\pi/2$ to $3\pi/4$, r becomes negative numerically increasing from 0 to ∞. So, during this range of θ, the point (r,θ) describes a curve in the fourth quadrant.



Again as θ increases from $\frac{3\pi}{4}$ to π r becomes positive

again and decreases from ∞ to 0, so that the point (r, θ) describes another part of the curve in the second quadrant.

The complete curve is shown in the diagram.

Illustrations :

Ex. 1. Find the area of the loop of the Folium of Descartes $x^3+y^3=3axy$. (C. H. 1968, 71)

By writing $x=r\cos\theta$, $y=r\sin\theta$, the equation of the curve in polar co-ordinates is

$$r=\frac{3a\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta}.$$

Since, the curve is symmetrical about the line $y=x$, as θ increases from 0 to $\frac{\pi}{4}$, half of the loop is described.

$$\therefore \quad \text{Area of the loop} = 2.\tfrac{1}{2}\int_0^{\pi/4} r^2 d\theta$$

$$=\int_0^{\pi/4}\frac{9a^2\sin^2\theta\cos^2\theta}{(\sin^3\theta+\cos^3\theta)^2}\,d\theta$$

$$=9a^2\int_0^{\pi/4}\frac{\tan^2\theta\,.\,\sec^2\theta}{(1+\tan^3\theta)^2}\,d\theta.$$

[Put $\tan\theta=z$
$\therefore$ $\sec^2\theta\,d\theta=dz$]

$$=9a^2\int_0^1\frac{z^2dz}{(1+z^3)^2}$$

$$=3a^2\left[-\frac{1}{1+z^3}+1\right]_0^1$$

$$=3a^2\left(-\frac{1}{2}+1\right)=\frac{3}{2}a^2.$$

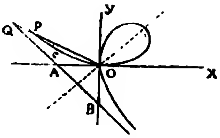
Ex. 2. Find the area bounded by the Folium of Descartes $x^3+y^3=3axy$, and its asymptotes. (C. H. 1957)

The equation of the asymptote of the folium is

$$x+y+a=0$$

and its polar equation is

$$r=\frac{-a}{\sin\theta+\cos\theta}. \quad \cdots \quad \cdots \quad (1)$$



The asymptote intersect the two axes $y=0$ and $x=0$ at A and B respectively whose co-ordinates are $(-a, 0)$, $(0, -a)$

$$\therefore \quad OA=a, \ OB=a$$

So that $\quad \Delta OAB=\frac{1}{2}a^2. \quad \cdots \quad \cdots \quad (2)$

Again from (1) as $r\to\infty$, $\sin\theta+\cos\theta\to 0$

i.e., $\tan\theta\to -1$ i.e., $\theta\to\frac{3\pi}{4}$.

$\therefore$ the radius vector at a point on the curve where it meets the asymptote will make an angle $\frac{3\pi}{4}$ with the x-axis.

If we now draw any line OP at an angle θ with OX cutting the curve and asymptote at P and Q respectively then clearly,

$$\frac{3\pi}{4}<\theta<\pi.$$

Let S_1 and S_2 denote the area of the triangle OQA and the area $OCPO$ respectively.

Then S_1=area bounded by the asymptote and the radius vectors varying from θ to π

$$=\frac{1}{2}\int_{\theta}^{\pi} r^2\, d\theta$$

$$=\frac{1}{2}\int_{\theta}^{\pi}\frac{a^2}{(\sin\theta+\cos\theta)^2}\, d\theta$$

$$=\tfrac{1}{2}a^2\int_{\theta}^{\pi}\frac{\sec^2\theta}{(1+\tan\theta)^2}\,d\theta$$

[Put $1+\tan\theta=z$
$\therefore\ \sec^2\theta\,d\theta=dz$]

$$=\tfrac{1}{2}a^2\int_{1+\tan\theta}^{1}\frac{dz}{z^2}$$

$$=\tfrac{1}{2}a^2\left[-\frac{1}{z}\right]_{1+\tan\theta}^{1}$$

$$=\frac{1}{2}a^2\left[\frac{1}{1+\tan\theta}-1\right] \quad \cdots \quad \cdots \quad (3)$$

Also, S_2=area bounded by the curve and the radius vector varying from θ to π

$$=\frac{1}{2}\int_{\theta}^{\pi}r^2\,d\theta \quad =\frac{1}{2}\int_{\theta}^{\pi}\frac{9a^2\sin^2\theta\cos^2\theta\,d\theta}{(\sin^3\theta+\cos^3\theta)^2}$$

$$=\frac{9a^2}{2}\int_{\theta}^{\pi}\frac{\tan^2\theta\sec^2\theta}{(1+\tan^3\theta)^2}\,d\theta.$$

[Put $1+\tan^3\theta=z$ $\therefore$ $3\tan^2\theta\sec^2\theta\,d\theta=dz$]

$$=\frac{9a^2}{2}\cdot\frac{1}{3}\int_{1+\tan^3\theta}^{1}\frac{dz}{z^2} \qquad =\frac{3a^2}{2}\left[-\frac{1}{z}\right]_{1+\tan^3\theta}^{1}$$

$$=\frac{3a^2}{2}\left[\frac{1}{1+\tan^3\theta}-1\right].$$

$\therefore$ Area $OCPQAO=S_1-S_2$

$$=\frac{a^2}{2}\left[\frac{1}{1+\tan\theta}-1\right]-\frac{3a^2}{2}\left[\frac{1}{1+\tan^3\theta}-1\right]$$

$\therefore$ Area included between the curve, the asymptote and the x axis (in the 2nd quadrant)

$$=\underset{\theta\to\frac{3\pi}{4}}{Lt}(S_1-S_2)$$

$$=\underset{\theta\to\frac{3\pi}{4}}{Lt}\frac{a^2}{2}\left[\frac{1}{1+\tan\theta}-1-\frac{3}{1+\tan^3\theta}+3\right]$$

$$=\underset{\theta\to\frac{3\pi}{4}}{Lt}\cdot\frac{a^2}{2}\left[2+\frac{1}{1+\tan\theta}-\frac{3}{1+\tan^3\theta}\right]$$

$$=\underset{\theta\to\frac{3\pi}{4}}{Lt}\cdot\frac{a^2}{2}\left[2+\frac{1-\tan\theta+\tan^2\theta-3}{1+\tan^3\theta}\right]$$

$$=\frac{a^2}{2}\cdot\underset{\theta\to\frac{3\pi}{4}}{Lt}\left[2+\frac{\tan^2\theta-\tan\theta-2}{1+\tan^3\theta}\right]$$

$$=\frac{a^2}{2}\cdot\underset{\theta\to\frac{3\pi}{4}}{Lt}\left[2+\frac{(\tan\theta+1)(\tan\theta-2)}{(1+\tan\theta)(1-\tan\theta+\tan^2\theta)}\right]$$

$$=\frac{a^2}{2}\underset{\theta\to\frac{3\pi}{4}}{Lt}\left[2+\frac{\tan\theta-2}{1-\tan\theta+\tan^2\theta}\right].$$

$$=\frac{a^2}{2}.$$

Also, from symmetry the similar area in the 4th quadrant

$$=\frac{a^2}{2}.$$

Hence, the required area included between the curve and the asymptote

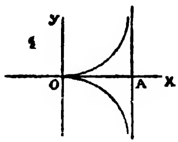
$$=\frac{a^2}{2}+\frac{a^2}{2}+\frac{a^2}{2}=\frac{3a^2}{2}.$$

Thus, area betwen the curve and asymptote is equal to the area of its loop.

15.7. Cissoid of Diocles.

The equation of the Cissoid is

$$y^2(2a-x)=x^3.$$



and its polar equation is

$$r=\frac{2a\sin^2\theta}{\cos\theta}.$$

It has a cusp at the origin with a tangent $y=0$ i.e., x-axis.

The curve is symmetrical with respect to x-axis and has an asymptote $x=2a$, so that in the diagram $OA=2a$.

Ex. 1. Find the area between the curve

$$\mathbf{x^3=y^2(a-x),\ a>0}$$

and its asymptote. (C. H. 1958)

The curve is a Cissoid of Diocles passing through the origin. It has a cusp at the origin with a tangent $y=0$ i.e., x-axis. Hence, the curve is symmetrical with respect to x axis.

$$\therefore\quad \text{Required area}=2\int_0^a y\,dx$$

$$=2\int_0^a \sqrt{\frac{x^3}{a-x}}\,dx.$$

$$=2\int_0^a x\sqrt{\frac{x}{a-x}}\,dx.$$

[Put $x=a\sin^2\theta$, $dx=2a\sin\theta\cos\theta\,d\theta$]

$$=2\int_0^{\pi/2} a\sin^2\theta\sqrt{\frac{a\sin^2\theta}{a\cos^2\theta}}.\ 2a\sin\theta\cos\theta\,d\theta$$

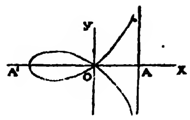
$$=4a^2\int_0^{\pi/2}\sin^4\theta\, d\theta=4a^2.\frac{3}{4}\frac{1}{2}.\frac{\pi}{2}=\frac{3}{4}\pi a^2 .$$

15.8. Strophoid.

The equation of the Strophoid is

$$y^2=x^2\frac{a+x}{a-x}.$$

As x varies from $-a$ to a, y is real. So, there is no curve for $x>a$ or for $x<-a$. For $-a\leqslant x\leqslant 0$, y has two values equal and opposite forming a loop. Origin is a node with two tangents $y=\pm x$. From 0 to a the curve extends to meet the asymptote $x=a$.



Illustrations :

Ex. Find the area included between the strophoid

$$y^2=x^2\frac{a+x}{a-x}$$

and its asymptote.

Since, the curve is symmetrical about the x axis

$$A=2\int_0^a y\, dx=2\int_0^a x\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}} dx$$

[Put $x=a\cos\theta$, $dx=a\sin\theta\, d\theta$]

$$=2\int_{\pi/2}^{0} a\cos\theta\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{\frac{1}{2}}(-a\sin\theta)\, d\theta$$

$$=2a^2\int_0^{\pi/2}\cos\theta.\ \cot\frac{\theta}{2}.\sin\theta\, d\theta$$

$$=2a^2\int^{\pi/2}\cos\theta(1+\cos\theta)\, d\theta$$

$$=2a^2\int_0^{\pi/2}(\cos\theta+\cos^2\theta)\,d\theta$$

$$=2a^2\Big[\sin\theta\Big]_0^{\pi/2}+2a^2\cdot\frac{\pi}{4}$$

$$=2a^2+\frac{\pi a^2}{2}=2a^2\left(1+\frac{\pi}{4}\right).$$

Ex. 2. Show that the area between the Folium of Descartes and its asymptotes is equal to the area of its loop.

Alternative to Ex. 2, Art 15.6.

Equation of the Folium of Descartes is

$$x^3+y^3=3axy.$$

Let the axes be rotated through are angle $\pi/4$ then writing $\frac{1}{\sqrt2}(x-y)$ and $\frac{1}{\sqrt2}(x+y)$ for x, y the transformed equation becomes

$$\frac{1}{2\sqrt2}(x-y)^3+\frac{1}{2\sqrt2}(x+y)^3=3a\cdot\frac{1}{2}(x^2-y^2)$$

or, $(x-y)^3+(x+y)^3=3\sqrt2 a(x^2-y^2)$

or, $2x^3+6xy^2=3\sqrt2\,ax^2-3\sqrt2 ay^2$

or, $y^2(6x+3\sqrt2 a)=x^2(3\sqrt2 a-2x)$

or, $y^2\left(x+\frac{1}{\sqrt2}a\right)=x^2\left(\frac{1}{\sqrt2}a-\frac{1}{3}x\right)$

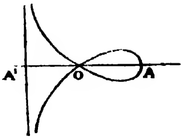
or, $y^2(x+c)=x^2(c-\frac{1}{3}x)$ [Writing $c=a/\sqrt2$]

or, $y^2=x^2\dfrac{3c-x}{3(c+x)}$

This is an inverted Strophoid with one loop between $x=0$ and $x=3c$ and an asymptote $x+c=0$. i.e. $x=-c$.

Here $y=\dfrac{1}{\sqrt3}\dfrac{x(3c-x)}{\sqrt{(c+x)(3c-x)}}$

Area of the loop $=2\int_0^{3c} y\,dx$



$$=\frac{2}{\sqrt{3}}\int_0^{3c}\frac{x(3c-x)}{\sqrt{(c+x)(3c-x)}}\,dx$$

[Put $x=c-2c\cos\theta,\ dx=2c\sin\theta\,d\theta$]

$$\left[\therefore\ I=\frac{2}{\sqrt{3}}\int\frac{(c-2c\cos\theta)(2c+2c\cos\theta)\,.\,2c\sin\theta\,d\theta}{\sqrt{(2c-2c\cos\theta)(2c+2c\cos\theta)}}\right.$$

$$=\frac{2c}{\sqrt{3}}\cdot\int\frac{(1-2\cos\theta)(1+\cos\theta)}{\sqrt{1-\cos^2\theta}}\,.\,2c\sin\theta\,d\theta$$

$$=\frac{4}{\sqrt{3}}c^2\int(1+\cos\theta)(1-2\cos\theta)\,d\theta$$

$$=-\frac{4}{\sqrt{3}}c^2\int(\cos\theta+\cos 2\theta)\,d\theta$$

$$\left.=-\frac{4}{\sqrt{3}}c^2(\sin\theta+\tfrac{1}{2}\sin 2\theta)\right]$$

$$=-\frac{4}{\sqrt{3}}c^2\left\{\sin\left(\cos^{-1}\frac{c-x}{2c}\right)+\tfrac{1}{2}\sin\left(2\cos^{-1}\frac{c-x}{2c}\right)\right\}_0^{3c}$$

......(1)

$$=-\frac{4}{\sqrt{3}}c^2\left\{(\sin\pi+\tfrac{1}{2}\sin 2\pi)-\left(\sin\frac{\pi}{3}+\tfrac{1}{2}\sin\frac{2\pi}{3}\right)\right\}$$

$$=\frac{4}{\sqrt{3}}c^2\left\{\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{4}\right\}=\frac{4c^2}{\sqrt{3}}\cdot\frac{3\sqrt{3}}{4}$$

$$=3c^2=\frac{3a^2}{2}.$$

The area included between the folium and the asymptote

$$=2\int^{-c} y\,dx \qquad \because\quad OA'=-c.$$

$$=\frac{2}{\sqrt{3}}\int^{-c}\frac{x(3c-x)}{\sqrt{(c+x)(3c-x)}}\,dx.$$

$$= \frac{2}{\sqrt{3}} \underset{\epsilon \to c}{Lt} \int_0^{\epsilon} \frac{x(3c-x)dx}{\sqrt{(c+x)(3c-x)}}$$

$$= -\frac{4}{\sqrt{3}} c^2 . \underset{\epsilon \to c}{Lt} \left[\sin\left(\cos^{-1}\frac{c-x}{2c}\right) + \frac{1}{2} \sin\left(2\cos^{-1}\frac{c-x}{2c}\right)\right] \quad \text{by (1)}$$

$$= -\frac{4}{\sqrt{3}} c^2 \left\{ -\left(\sin\frac{\pi}{3} + \tfrac{1}{2}\sin\frac{2\pi}{3}\right)\right\}$$

$$= \frac{4}{\sqrt{3}} c^2 \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right) = \frac{4}{\sqrt{3}} c^2 . \frac{3\sqrt{3}}{4}$$

$$= 3c^2$$

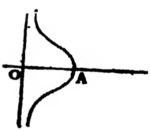
$$= 3 . \frac{a^2}{2}.$$

15.9. Witch of Agnesi.

The characteristics of this curve was first analysed by a Lady Italian Mathematician Maria Jactana Agnesi, Professor of Mathematics at Bologna.

The equation of the curve is

$$xy^2 = 4a^2(2a - x)$$



where $x=0$ is the only asymptote.

The curve is symmetrical about the x axis.

There is no part of the curve for $x > 2a$ and for the asymptote and the curve $0 < x \leqslant 2a$.

Ex. Find the area included between the curve

$$xy^2 = 4a^2(2a - x),\ a > 0$$

and its asymptote.

We have $y = \pm \frac{2a}{\sqrt{x}} \sqrt{2a - x}$.

Since, the curve is symmetrical about the x axis

$$A = 2\int_0^{2a} \frac{2a}{\sqrt{x}} \sqrt{2a-x}\, dx \quad \because \quad OA = 2a.$$

[Put $x = 2a \sin^2\theta$, $dx = 4a \sin\theta \cos\theta\, d\theta$]

$$=2\int_0 \frac{2a}{\sqrt{2a}\sin\theta}\sqrt{2a(1-\sin^2\theta)}\,4a\sin\theta\cos\theta\,d\theta$$

$$=16a^2\int_0^{\pi/2}\cos^2\theta\,d\theta=16a^2.\frac{\pi}{4}=4\pi a^2$$

15.10. Lemniscate of Bernoulli. (C. H. 1968)

The equation of the Lemniscate is

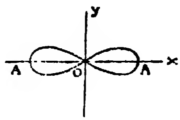
$$(x^2+y^2)^2=a^2(x^2-y^2).$$

Putting $x=r\cos\theta$, $y=r\sin\theta$, its polar equation becomes

$$(r^2\cos^2\theta+r^2\sin^2\theta)^2=a^2(r^2\cos^2\theta-r^2\sin^2\theta)$$

or, $r^2=a^2\cos 2\theta$.

The curve has two symmetrical loops with a node at the origin, $y=\pm x$ being the two tangents at the origin.



Illustrations :

Ex. 1. Find the area of the two loops of (C. H. 1968)

$$r^2=a^2\cos 2\theta.$$

The curve is a Lemniscate and so has two loops.

Since, each loop is symmetrical about the x axis, half of a loop is described when θ varies from 0 to $\frac{\pi}{4}$.

$\therefore$ Area of the two loops

$$=4.\tfrac{1}{2}\int_0^{\pi/4} r^2 d\theta$$

$$=2\int_0^{\pi/4} a^2\cos 2\theta\,d\theta=2a^2\Big[\tfrac{1}{2}\sin 2\theta\Big]_0^{\pi/4}=a^2$$

Ex. 2. Find the whole area of the curve

$$\mathbf{a^2y^4=x^4(a^2-x^2).}$$

Writing $x = r\cos\theta$, $y = r\sin\theta$, the polar equation becomes

$$a^2 r^4 \sin^4\theta = r^4 \cos^4\theta (a^2 - r^2\cos^2\theta)$$

or, $a^2 \sin^4\theta = a^2\cos^4\theta - r^2\cos^6\theta$

or, $r^2 \cos^6\theta = a^2(\cos^4\theta - \sin^4\theta)$

or, $r^2 = a^2 \dfrac{\cos 2\theta}{\cos^6\theta}$.

The curve consists of two loops with a node at the origin. The tangent at the origin being $y = \pm x$.

As θ varies from 0 to $\pi/4$, half the area of a loop is described.

$$\therefore \quad \text{Total area} = 4\int_0^{\pi/4} r^2\, d\theta = 4\int_0^{\pi/4} a^2 \frac{\cos 2\theta}{\cos^6\theta}\, d\theta$$

$$\therefore \quad = 4a^2 \int_0^{\pi/2} \frac{\cos^2\theta - \sin^2\theta}{\cos^6\theta} d\theta$$

$$= 4a^2 \int_0^{\pi/4} \frac{\cos^4\theta - \sin^4\theta}{\cos^6\theta} d\theta$$

$$= 4a^2 \int_0^{\pi/4} \sec^2\theta\, d\theta - 4a^2 \int_0^{\pi/4} \tan^4\theta \sec^2\theta\, d\theta$$

$$= 4a^2 \Big[\tan\theta\Big]_0^{\pi/4} - 4a^2 \int_0^1 z^4\, dz \qquad \left[\begin{array}{l}\text{Put } \tan\theta = z \\ \sec^2\theta\, d\theta = dz\end{array}\right]$$

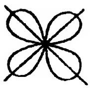
$$= 4a^2 - 4a^2 . \tfrac{1}{5} = \tfrac{16}{5} a^2.$$

15.11. Rose-Petals.

The type of curves represented by

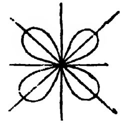
$r = a\cos n\theta$, $r = a\sin n\theta$ when n is a positive integer are known as Rose-petal curves.

If n be even there are $2n$ loops in the curves.



(fig. 1)

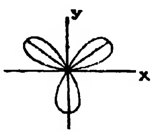
$r=a \cos 2\theta$



(fig. 2)

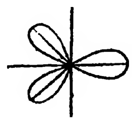
$r=a \sin 2\theta$

Thus if $n=2$, the curves of $r=a \cos 2\theta$ and $r=a \sin 2\theta$ have four equal loops all lying on a circle with centre at the origin and radius equal to a. The loop of the first curve lie between the tangents $y=\pm x$ and that of the 2nd lie between $x=0$, $y=0$.



(fig. 4)

$r=a \sin 3\theta$



(fig. 3)

$r=a \cos 3\theta$

If n be odd$=3$, each curve contains three loops all lying on a circle with centre at the origin and radius equal to a.

In $r=a \sin 3\theta$, the loops lie in the interval

$$0 \leqslant \theta \leqslant \frac{\pi}{3}, \quad \frac{2\pi}{3} \leqslant \theta \leqslant \pi, \quad \frac{4}{3}\pi \leqslant \theta \leqslant \frac{5\pi}{3}.$$

Ex. 1. Find the area of the loops of (C. H. 1967)

$$\mathbf{r=a \sin 3\theta, \ a>0.}$$

The loop in the Ist quadrant lies between $0 \leqslant \theta \leqslant \frac{\pi}{3}$.

$\therefore$ Area of this loop$=\frac{1}{2}\int^{\pi/3} r^2 d\theta$

$$=\frac{1}{2}\int_0^{\pi/3} a^2 \sin^2 3\theta \, d\theta$$

[Put $3\theta = z$, $d\theta = \frac{1}{3}dz$]

$$=\frac{a^2}{6}\int_0^{\pi} \sin^2 z \, dz$$

$$=\frac{a^2}{12}\int_0^{\pi} (1-\cos 2z) \, dz$$

$$=\frac{a^2}{12}\left[z-\frac{\sin 2z}{2}\right]_0^{\pi}$$

$$=\frac{1}{12}a^2\pi.$$

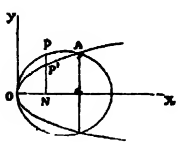
$\because$ There are 3 loops in the curve

$\therefore$ Total area $=3\times\frac{1}{12}a^2\pi=\frac{1}{4}a^2\pi$.

Miscellaneous Examples :

Ex. 1. Find the area which lies in the first quadrant and is bounded by the circle $x^2+y^2=2ax$ and the parabola $y^2=ax$. **(C.H. 1960)**

The two curves meet in the 1st quadrant at $x=0$ and $x=a$. Also, the two curves are symmetrical about the x axis.



Required area = Area $OP'APO$

=(Area included between the circle, the x axis and the ordinates at $x=0$, $x=a$)$-$(Area included between the parabola, the x axis and the ordinates at $x=0$, $x=a$).

$$=\int_0^a \sqrt{2ax-x^2}\,dx-\int_0^a \sqrt{ax}\,dx$$

For the 1st integral, put $x=2a\sin^2\theta$

$\therefore$ For 1st integral $dx = 4a \sin\theta \cos\theta \, d\theta$

$$= \int_0^{\pi/4} \sqrt{4a^2 \sin^2\theta - 4a^2 \sin^4\theta} \cdot 4a \sin\theta \cos\theta \, d\theta$$

$$= 8a^2 \int_0 \sin^2\theta \cos^2\theta \, d\theta = 2a^2 \int_0 \sin^2 2\theta \, d\theta$$

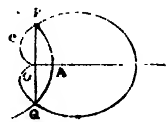
$$= a^2 \int_0^{\pi/4} (1 - \cos 4\theta) \, d\theta = a^2 \left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/4} = \frac{a^2\pi}{4}$$

$$\text{2nd Integral} = \sqrt{a} \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a = \tfrac{2}{3} \sqrt{a} \cdot a^{\frac{3}{2}} = \tfrac{2}{3} a^2$$

$$\therefore \quad \text{Required Area} = \tfrac{1}{4}\pi a^2 - \tfrac{2}{3} a^2 = a^2 \left(\frac{\pi}{4} - \frac{2}{3}\right).$$

Ex. 2. Find the area enclosed by the curve
$r = 2a(1 + \cos\theta)$ and $r = 2a/(1 + \cos\theta)$.

Solving the two equations, the polar co-ordinates of the point of intersection are



$$P\left(2a, \frac{\pi}{2}\right) \text{ and } Q\left(2a, \frac{3\pi}{2}\right).$$

Since, the two curves are symmetrical about the initial line.

Required Area $= 2$. Area $OAPC$

$= 2[\text{Area } PCO + \text{Area } AOP]$

$$= 2\left[\tfrac{1}{2} \int_{\pi/2}^{\pi} r^2 d\theta \text{ for the cardioide} + \tfrac{1}{2} \int_0^{\pi/2} r^2 d\theta \text{ for the parabola}\right.$$

$$= \int_{\pi/2}^{\pi} 4a^2 (1 + \cos\theta)^2 \, d\theta + \int_0^{\pi/2} \frac{4a^2}{(1 + \cos\theta)^2} \, d\theta = I_1 + I_2 \text{ (say)}$$

$$I_1=4a^2\int_{\pi/2}^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$=4a^2\int_{\pi/2}^{\pi}\{1+2\cos\theta+\tfrac{1}{2}(1+\cos 2\theta)\}\,d\theta$$

$$=4a^2\Big[\theta+2\sin\theta+\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta\Big]_{\pi/2}^{\pi}$$

$$=4a^2\Big[(\pi+\tfrac{1}{2}\pi)-\Big(\frac{\pi}{2}+2+\frac{\pi}{4}\Big)\Big]=4a^2\Big(\frac{3\pi}{4}-2\Big)$$

$$I_2=4a^2\int_0^{\pi/2}\frac{d\theta}{(1+\cos\theta)^2}=4a^2\int_0^{\pi/2}\frac{d\theta}{4\cos^4\frac{\theta}{2}}$$

$$=a^2\int_0^{\pi/2}\sec^4\frac{\theta}{2}\,d\theta$$

$$=a^2\int_0^{\pi/2}\Big(1+\tan^2\frac{\theta}{2}\Big)\sec^2\frac{\theta}{2}\,d\theta$$

$$\Big[\text{Put }\tan\frac{\theta}{2}=z,\ \tfrac{1}{2}\sec^2\frac{\theta}{2}d\theta=dz\Big]$$

$$=2a^2\int_0^1(1+z^2)\,dz$$

$$=2a^2\Big[z+\frac{z^3}{3}\Big]_0^1=2a^2.\tfrac{4}{3}=\tfrac{8}{3}a^2.$$

$$\therefore\quad \text{Required Area}=4a^2\Big(\frac{3\pi}{4}-2\Big)+\frac{8}{3}a^2$$

$$=4a^2\Big(\frac{3\pi}{4}-2+\tfrac{2}{3}\Big)$$

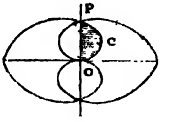
$$=4a^2\Big(\frac{3\pi}{4}-\tfrac{4}{3}\Big)$$

$$=\tfrac{1}{3}a^2(9\pi-16).$$

Ex. 8. Find the area of the region included between the Cardioides

$$\mathbf{r=a(1+\cos\theta) \text{ and } r=a(1-\cos\theta).}$$

(C. H. 1966)



Area of the region OCP

= Area included between the curve $r=a(1-\cos\theta)$ and radius vectors from 0 to $\frac{\pi}{2}$

$$=\frac{1}{2}\int_{0}^{\pi/2} r^2 d\theta$$

$$=\frac{1}{2}\int_{0}^{\pi/2} a^2(1-\cos\theta)^2 d\theta$$

$$=\frac{1}{2}a^2\int_{0}^{\pi/2}(1-2\cos\theta+\cos^2\theta)d\theta$$

$$=\frac{1}{2}a^2\Big[\theta-2\sin\theta\Big]_{0}^{\pi/2}+\frac{1}{2}a^2\cdot\frac{\pi}{4}$$

$$=\frac{1}{2}a^2\left[\frac{\pi}{2}-2+\frac{\pi}{4}\right]$$

$$=\frac{1}{2}a^2\left[\frac{3\pi}{4}-2\right].$$

$\therefore$ Required Area $=4\times\frac{1}{2}a^2\cdot\frac{1}{4}(3\pi-8)$

$$=\frac{1}{2}a^2(3\pi-8).$$

Exercise 15

1. If the length of the arc measured from the origin varies as the square root of the ordinate, show that the intrinsic equation of the curve is

$$S=4a\sin\psi.$$

Hence, show that the curve is a cycloid.

2. If $S=c\tan\psi$ be the intrinsic equation of the curve show that its cartesian equation is

$$y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right) \text{ provided } x=0 \text{ and } y=c \text{ when } \psi=0.$$

3. Find the intrinsic equation of the catenary

$$y=c\cosh\frac{x}{c}.$$

Hence, show that $y^2=c\rho=c^2+S^2$ the arc being measured from the vertex.

4. Find the intrinsic equation of the astroid

$$x=a\cos^3\theta,\ y=a\sin^3\theta.$$

Hence, show that if an arc S of the curve is measured from the point for which $x=0$, then

(i) $S\propto x^{\frac{2}{3}}$

(ii) $\rho^2+4S^2=6aS$.

5. Find the intrinsic equation of the cycloid

$$x=a(\theta+\sin\theta),\ y=a(1-\cos\theta).$$

Hence, show that $\rho^2+S^2=16a^2$.

The arc being measured from the vertex when $\theta=0$.

6. Show that the curve whose equation is $r=2a\cos^2\frac{\theta}{2}$ is a cardioide. Find its intrinsic equation and hence show that $S^2+9\rho^2=16a^2$, the arc being measured from the vertex where $\theta=0$.

7. Show that the intrinsic equation of the curve

$$x=e^t\sin t,\ y=e^t\cos t$$

is $Se^{-\frac{\pi}{4}}+\sqrt{2}=\sqrt{2}(\cosh\psi-\sinh\psi)$

where $\theta=\frac{\pi}{4}$ is the fixed poin

8. Show that the intrinsic equation of the equiangular spiral $r=ce^{\theta\cot\alpha}$ with $(c, 0)$ as the fixed point is $S=c\sec\alpha\{e^{\psi\cot\alpha}-1\}$.

9. Show that the intrinsic equation of the tractrix

$$x=a\cos\theta+a\log\tan\frac{\theta}{2},\ y=a\sin\theta$$

with $\theta=\frac{\pi}{2}$ as the fixed point is the curve $S=a\log\cos\psi$.

10. Show that the intrinsic equation of the curve

$$y=c\log\sec\left(\frac{x}{c}\right)$$

with origin as the fixed point is $S=c\log(\tan\psi+\sec\psi)$.

11. Show that the parametric equation

$$x=a\cos t\,(1-\cos t),\ y=a\sin t\ (1-\cos t)$$

represent the equation of a cardioide of area $\frac{3}{2}\pi a^2$.

12. Show that the curve $(x^2+y^2)^2=a^2(x^2-y^2)$ is a Lemniscate o Bernoulli and that the area enclosed by it is a^2.

13 Show that the area of the loop of the curve

$$x=\frac{1-t^2}{1+t^2},\ y=\frac{t-t^3}{t+t^2},\ -1\leqslant t\leqslant 1$$

is $(2-\pi/2)$.

14. Show that the area of the loop formed by the $x=t-t^3$, $y=1-t^4$ i 16/35.

15. Show that the area enclosed by the curve $a^2x^2=y^3(2a-y)$ is equal t the area of a circle of radius a.

16. Show that the area of a loop of the curve

$$r=a\frac{\sin\theta}{\cos\theta}$$

is $\frac{1}{8}a^2(9\sqrt{3}-4\pi)$.

17. Show that the area included between the two curves $r=a\sin 2\theta$ and $r=a\sin\theta$ is $\frac{a^2}{16}(4\pi-3\sqrt{3})$.

18. Show that the curve $r=a\left(1+2\sin\frac{\theta}{2}\right)$ has three loops, andt hat area of each loop is $a^2(8\pi+8)$, $\frac{1}{2}a^2(3\pi-8)$, $\frac{1}{2}a^2(3\pi-8)$ respectively.

19. Show that the arc of the upper half of the cardioide $r=a(1-\cos\theta)$ is bisected at $\theta=2\pi/3$.

20. Show that the intrinsic equation of the curve $y=\log x$, taking (1, 0) is the fixed point is $S(\cos\psi-\sin\psi)=\sqrt{2}$.

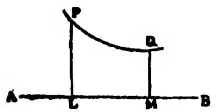
21. Show that the length of the curve $y=e^x$ measured from (0, 1) is

$$\sqrt{(e^2+1)}-\log\frac{1+\sqrt{1+e^2}}{\sqrt{2}+1}-(\sqrt{2}-1).$$

VOLUMES AND SURFACES OF REVOLUTION

16.1. Let P and Q be any two points on the curve $y=f(x)$ and PL, QM are drawn perpendicular to any line AB on the plane of the curve. Let the curve $y=f(x)$ does not intersect the line AB.

If now the region $PLMQ$ is allowed to revolve about the line AB, then a solid in space, with AB as its axis, will be generated. The solid so generated by the revolution of the curve PQ is called the volume of revolution. The line AB about which the curve rotates is called the axis of revolution.



Theorem :

16.2. The volume V obtained by revolving about the axis of x, the arc of the curve $y=f(x)$ intercepted between the two points on it whose abscissae are a, b is

$$V=\pi\int_a^b y^2 dx \quad \text{or,} \quad \pi\int_a^b \{f(x)\}^2 dx$$

it being assumed that the curve does not cross the x axis

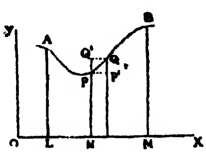


Fig. I

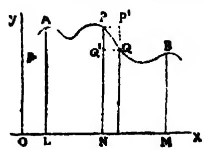


Fig. II

Let A and B be any two fixed points on the curve $y=f(x)$ whose abscissae are $x=a$ and $x=b$ respectively AL and BM are drawn perpendicular to the x axis.

We are to find the volume generated by the revolution of the arc APQ about the x axis.

Let $P(x, y)$ be any variable point on the curve so that the volume V generated by the revolution of the arc AP about the x axis is a function of x.

Let Q $(x+\delta x,\ y+\delta y)$ be a point on the curve so near to P that as a point moves from P to Q along the curve, the ordinate either increases as in Fig. I or decreases as in Fig. II.

Through P and Q draw two ordinates and complete the rectangular strips by drawing PP' and QQ'.

Then, the volume of the disc generated by revolving PP' about x axis

$$=\pi y^2.\delta x.$$

And the volume of the disc generated by revolving QQ' about the x axis

$$=\pi(y+\delta y)^2\delta x.$$

If now δV denote the volume generated by the revolution of the arc PQ about x axis then in Fig. I

$$\pi y^2\delta x<\delta V<\pi(y+\delta y)^2\delta x$$

or, $$\pi y^2<\frac{\delta V}{\delta x}<\pi(y+\delta y)^2$$

$$\Rightarrow\quad \frac{dV}{dx}=\pi y^2.$$

Also in Fig. II

$$\pi(y+\delta y)^2\delta x<\delta V<\pi y^2.\delta x$$

or, $$\pi(y+\delta y)^2<\frac{\delta V}{\delta x}<\pi y^2$$

$$\Rightarrow\quad \frac{dV}{dx}=\pi y^2$$

$$\Rightarrow \quad V=\pi \int_a^b y^2 dx \qquad \cdots \qquad \cdots \qquad (1)$$

$$\text{or,} \quad =\pi \int_a^b \{f(x)\}^2 dx$$

Cor. 1. The volume generated by the revolution of the arc of a curve $x=f(y)$ about y axis, intercepted between any two points on the curve where ordinates are

$$y=c \text{ and } y=d \text{ is } \pi \int_c^d x^2 dy, \qquad \cdots \qquad (2)$$

it being assumed that the curve does not cross the y axis.

Cor. 2. The formula (1) may be put in the form

$$V=\pi \int_{OL}^{OM} (PN)^2 d(ON)$$

Illustrations :

Ex. 1. The circle $x^2+y^2=a^2$ revolves about x axis. Find the volume of the sphere so generated.

As the upper part of the circle revolves about the x axis from $x=-a$ to a, the whole volume of the sphere is generated by one complete revolution.

$$\therefore \quad V=\pi \int_{-a}^{a} y^2\, dx=\pi \int_{-a}^{a} (a^2-x^2)\, dx$$

$$=\pi\left[a^2x-\frac{x^3}{3}\right]_{-a}^{a}=\pi\left[\left(a^3-\frac{a^3}{3}\right)-\left(-a^3+\frac{a^3}{3}\right)\right]$$

$$=\pi[\tfrac{2}{3}a^3+\tfrac{2}{3}a^3]=\tfrac{4}{3}\pi a^3.$$

Ex. 2. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ revolves about the major axis. Find the volume of the Prolate Spheroid so formed.

The Prolate Spheroid is generated by a complete revolution of the upper part of the ellipse about the major axis from $x=-a$ to a.

$$\therefore \quad V=\pi\int_{-a}^{a} y^2\,dx=\pi\int_{-a}^{a} b^2\left(1-\frac{x^2}{a^2}\right)dx$$

$$=\pi b^2\left[x-\frac{x^3}{3a^2}\right]_{-a}^{a}=\pi b^2\left[\left(a-\frac{a}{3}\right)-\left(-a+\frac{a}{3}\right)\right]$$

$$=\pi b^2\left[\tfrac{2}{3}a+\tfrac{2}{3}a\right]=\tfrac{4}{3}\pi ab^2.$$

Ex. 3. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ revolves about the minor axis. Find the Oblate Spheroid so formed.

As the curve on the right side of the y axis revolves about y axis, the Oblate Spheroid is generated.

$$\text{So,}\quad V=\pi\int_{-b}^{b} x^2\,dy.$$

$$=\pi\int_{-b}^{b} a^2\left(1-\frac{y^2}{b^2}\right)dy$$

$$=\pi a^2\left[y-\frac{y^3}{3b^2}\right]_{-b}^{b}$$

$$=\tfrac{4}{3}\pi a^2 b.$$

Ex. 4. The sine curve $y = \sin x$ revolves about the x axis. Find the volume so generated from $x = 0$ and $x = \pi$.

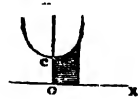
$$V = \pi \int_0^\pi y^2\, dx$$

$$= \pi \int^\pi \sin^2 x dx = \frac{\pi}{2}\int^\pi 2 \sin^2 x\, dx$$

$$= \tfrac{1}{2}\pi \int_0^\pi (1 - \cos 2x)\, dx = \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{\pi^2}{2}.$$

Ex. 5. The area bounded by the catenary

$y = a \cosh \frac{x}{a}$, the x axis, the y axis and an ordinate at (x, y) revolves about x axis. Find the volume of the Cetenoid so formed in terms of x, y and S where S the length of the curve from the vertex to the point (x, y).



$$V = \pi \int_0^x y^2 dx$$

$$= \pi \int_0^x a^2 \cosh^2 \frac{x}{a}\, dx$$

$$= \frac{\pi a^2}{2}\int_0^x \left(1 + \cosh \frac{2x}{a}\right) dx = \frac{\pi a^2}{2}\left[x + \frac{a}{2} \sinh \frac{2x}{a}\right]^x$$

$$= \frac{\pi a^2}{2}\left[x + a \sinh \frac{x}{a} \cosh \frac{x}{a}\right]^x.$$

If S be the length of the Catenary from $(0, a)$ to the point (x, y).

Then $S = a \sinh \dfrac{x}{a}$

$$\therefore \quad V = \frac{\pi a^2}{2}\left\{x + S\left(\frac{y}{a}\right)\right\}$$

$$= \frac{\pi a}{2}(ax + Sy).$$

Ex. 6. Find the volume of the solid generated by revolving the cycloid

$$x = a(\theta + \sin\theta), \quad y = a(1 + \cos\theta)$$

about its base. (C. H. 1964, 71)

From the given equation, it follows that the base of the cycloid is the x axis. One arc of the cycloid extends from $x = -a\pi$ to $a\pi$

So, the extreme values of θ are $\theta = \pm\pi$

$$\therefore \quad V = \pi\int_{-a\pi}^{a\pi} y^2 dx$$

$$= \pi a^3 \int_{-\pi}^{\pi} (1 + \cos\theta)^3 \, d\theta$$

$$= 8\pi a^3 \int_{-\pi}^{\pi} \cos^6 \frac{\theta}{2} \, d\theta \qquad \left.\text{Put } \frac{\theta}{2} = z,\ d\theta = 2dz\right]$$

$$= 16\pi a^3 \int_{\pi/2}^{-\pi/2} \cos^6 z \, dz$$

$$= 32\pi a^3 \int_{0}^{\pi/2} \cos^6 z \, dz$$

$$= 32\pi a^3 \cdot \frac{5}{6}\frac{3}{4}\frac{1}{2} \cdot \frac{\pi}{2}$$

$$= 5\pi^2 a^3$$

Ex. 7. The area included between the upper part of the the astroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ and the x axis, revolves about the x axis. Find the volume so generated.

Let the equation of the astroid be

$$x=a\cos^3\theta,\ y=a\sin^3\theta.$$

The volume generated by the revolution of the area in the first quadrant about x axis is $\frac{1}{2}$ of the total volume.

$$\therefore\quad V=2\pi\int_0^a y^2dx$$

$$=2\pi\int_{\pi/2}^{0} a^2\sin^6\theta\,.\,3a\cos^2\theta\,(-\sin\theta)\,d\theta$$

$$=6\pi a^3\int_0^{\pi/2}\sin^7\theta\cos^2\theta\,d\theta$$

$$=6\pi a^3\int_0^{\pi/2}\sin^7\theta\,(1-\sin^2\theta)\,d\theta$$

$$=6\pi a^3\int_0^{\pi/2}(\sin^7\theta-\sin^9\theta)\,d\theta$$

$$=6\pi a^3[\tfrac{6}{7}\,.\,\tfrac{4}{5}\,.\,\tfrac{2}{3}-\tfrac{8}{9}\,.\,\tfrac{6}{7}\,.\,\tfrac{4}{5}\,.\,\tfrac{2}{3}]$$

$$=\tfrac{32}{105}\pi a^3.$$

Ex. 8. Find the volume generated by the revolution of the Cissoid $y^2(2a-x)=x^3$ about its asymptotes.

Here $y=\dfrac{x^{\frac{3}{2}}}{\sqrt{2a-x}}$

$$\therefore\ dy=\frac{\frac{3}{2}x^{\frac{1}{2}}\sqrt{2a-x}+\dfrac{x^{\frac{3}{2}}}{2\sqrt{2a-x}}}{2a-x}\,dx$$

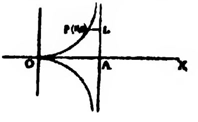
$$=\frac{3\sqrt{x}(2a-x)+x^{\frac{3}{2}}}{2(2a-a)^{\frac{3}{2}}}dx$$

$$=\frac{\sqrt{x}(6a-2x}{2\ 2a-x)^{\frac{3}{2}}}dx$$

$$=\frac{\sqrt{x}(3a-x)}{(2a-x)^{\frac{3}{2}}}.dx$$

$$=\frac{\sqrt{x}\ 3a-x)(2a-x)^{\frac{1}{2}}}{(2a-x)^2}dx \qquad (1)$$

Let $P(x, y)$ be any point on the curve in the first quadrant and draw PL perpendicular to the asymptote $x=2a$.



Then as $OA=2a$

$PL=2a-x$

Now, the volume generated by the revolution of the curve in the 1st quadrant about the asymptote is half the required volume.

$$\therefore \quad V=2\pi\int_0^{\infty}(PL)^2d(AL)$$

$$=2\pi\int_0^{\infty}(2a-x)^2dy$$

$$=2\pi\int_0^{2a}\frac{(2a-x)^2.\ \sqrt{x}(3a-x)(2a-x)^{\frac{1}{2}}}{(2a-x)^2}dx$$

[changing y to x by (1)]

$$=2\pi\int_0^{2a}(3a-x)\sqrt{x(2a-x)}\,dx.$$

[Put $x=2a\sin^2\theta$

$\therefore\ dx=4a\sin\theta\cos\theta\, d\theta$]

$$\therefore \quad V=16\pi a^3 \int^{\pi/2} (3-2\sin^2\theta)\sin^2\theta\cos^2\theta\, d\theta$$

$$=48\pi a^3 \int_0 \sin^2\theta\cos^2\theta\, d\theta-32\pi a^3 \int_0 \sin^4\theta\cos^2\theta\, d\theta$$

$$=48\pi a^3\left[\int^{\pi/2} \sin^2\theta\, d\theta-\int_0^{\pi/2} \sin^4\theta\, d\theta\right]$$

$$-32\pi a^3\left[\int_0^{\pi/2} \sin^4\theta\, d\theta-\int_0^{\pi/2} \sin^6\theta\, d\theta\right]$$

$$=48\pi a^3\left[\frac{\pi}{4}-\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]-32\pi a^3\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

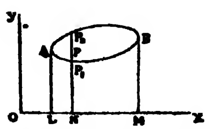
$$=48\pi a^3\cdot\frac{\pi}{4}\cdot\frac{1}{4}-32\pi a^3\left[\frac{3\pi}{16}\cdot\frac{1}{6}\right]$$

$$=3\pi^2 a^3-\pi^2 a^3=2\pi^2 a^3.$$

16.3. Pappus Theorem :

If a closed plane curve revolves about a straight line (not intersected by the curve) in its plane, then the volume of the solid generated is equal to the product of the area of the enclosed region and the length of the path described by the centroid of the region. (C. H. 1969)

Let the x axis be the axis of revolution and let the closed



curve be such that every ordinate between A and B parallel to y axis cut the curve in two and only two points. Let PN be such an ordinate with same abscissa x cutting the curve at P_1 and P_2 so that $P_1N=y_1$ and $P_2N=y_2$.

Then, y_1 and y_2 are functions of x varying from $x=a$ to $x=b$.

Now, the volume of solid generated by the revolution of the arcs AP_2B and AP_1B about the x axis are

$$\pi\int_a^b y_2^2dx \text{ and } \pi\int_a^b y_1^2dx$$

Hence, the volume of the solid generated by the revolution of the closed curve about x axis is

$$V=\pi\int_a^b y_2^2dx-\pi\int_a^b y_1^2dx$$

$$=\pi\int_a^b (y_2^2-y_1^2)\,dx. \qquad \cdots \quad (1)$$

Let $\bar{y}$ be the ordinate of the centroid of the region enclosed by the closed curve and A be the area of the region. Then

$$\bar{y}=\frac{\int_a^b \frac{1}{2}(y_1+y_2)(y_2-y_1)\,dx}{A}$$

so that $\int^b (y_2^2-y_1^2)\,dx=2A\bar{y}$ (2)

Hence, from (1)

$$V=2\pi\bar{y}A.$$

Illustrations :

Ex. 1. Find the volume generated by the revolution of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the line $x=2a$.

The area A of the ellipse is πab, perpendicular distance of C.G. of the ellipse which is at (0, 0) from the line $x=2a$ is $2a$.

$\therefore$ the length of the path described by $2a$ is $2\pi(2a)=4\pi a$.

Hence, by Pappus Theorem,

$V=$(Area revolved)$\times$(length of the path described by the C.G. of the area)

$=\pi ab \cdot 4\pi a$

$=4\pi^2 a^2 b.$

Ex. 2. Find the volume of the solid generated by the revolution of the plane area bounded by the circle $x^2+(y-b)^2=a^2$ about the x axis, it being assumed that $b>a$. **(C. H. 1969)**

The circle is of radius a. So its area $A=\pi a^2$. The centre of the circle is at $(0, b)$ on the y axis.

$\therefore$ The distance of C.G. from x axis$=b$ and the length of the path described by $b=2\pi b$.

Hence, by Pappus Theorem, required volume,

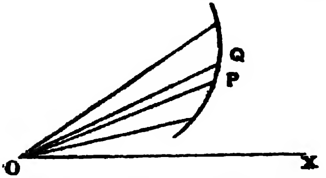
$V=$(Area revolved)$\times$(length of the path described by the C.G. of the area)

$=\pi a^2 . 2\pi b=2\pi^2 a^2 b.$

16.4. Volume of a solid formed by the rotation of an area bounded by the curve $r=f(\theta)$ and the lines $\theta=\alpha$, $\theta=\beta$; about the line $\theta=0$ is

$$V=\tfrac{2}{3}\pi \int_{\alpha}^{\beta} r^3 \sin \theta \, d\theta.$$

Let $P(r, \theta)$ be any point on the curve $r=f(\theta)$ and $Q(r+\delta r, \theta+\delta\theta)$ be any other point on the curve.



Then, the area of the triangle formed by POQ

$=\frac{1}{2}r(r+\delta r)\sin \delta\theta$

$=\frac{1}{2}r^2\delta\theta$ up to the 1st order.

The C.G. of this elementary area is at a distance $\frac{2}{3}r$ from O.

$\therefore$ perpendicular distance of C.G. from $OX=\frac{2}{3}r \sin \theta$

$\therefore$ By Pappus Theorem, the volume formed by the rotation of this elementary area about OX

$= $ (Area revolved) $\times$ (length of the path described by the C.G. of the area)

$$=\tfrac{1}{2}r^2 d\theta \times 2\pi(\tfrac{2}{3}r \sin \theta)$$

$$=\tfrac{2}{3}\pi r^3 \sin \theta \, d\theta.$$

Hence, integrating between the limits $\theta=\alpha$ and $\theta=\beta$, the total volume generated

$$=\tfrac{2}{3}\pi \int_{\alpha}^{\beta} r^3 \sin \theta \, d\theta$$

Illustrations :

Ex. 1. Find the volume of the solid generated by revolving the Cardioide $r=a(1-\cos \theta)$ about the initial line.

If the upper part of the cardioide be revolved about the initial line from $\theta=0$ to $\theta=\pi$ the volume of the solid of revolution generated is

$$V=\tfrac{2}{3}\pi \int r^3 \sin \theta \, d\theta$$

$$=\tfrac{2}{3}\pi \int_0^{\pi} a^3(1-\cos \theta)^3 \sin \theta \, d\theta$$

[Put $1-\cos \theta=z$, $\sin \theta \; d\theta=dz$]

$$=\tfrac{2}{3}\pi a^3 \int_0^2 z^3 dz=\tfrac{2}{3}\pi a^3 . \tfrac{16}{4}=\tfrac{8}{3}\pi a^3.$$

Ex. 2. The curve $r=a(1+\cos \theta)$ revolves about the initial line. Find the volume of the figure formed. (C.H. 1968)

Here $V=\frac{2}{3}\pi\int_0^{\pi} a^3(1+\cos\theta)^3 \sin\theta \, d\theta$

[Put $1+\cos\theta=z, -\sin\theta \, d\theta=dz$]

$$= -\tfrac{2}{3}\pi a^3 \int_2^0 z^3 \, dz$$

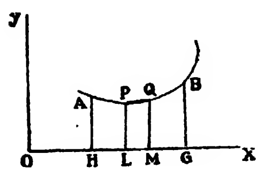
$$= \tfrac{2}{3}\pi a^3 \int_0^2 z^3 dz = \tfrac{8}{3}\pi a^3 \text{ as in Ex. 1.}$$

16.5. Surface of Revolution :

The area of the surface of the solid generated by revolving the arc of the curve $y=f(x)$ about the x axis between the limits whose abscissae are a and b is

$$2\pi \int_a^b y \frac{ds}{dx} \, dx \quad \text{i.e.} \quad 2\pi \int_a^b y \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, dx$$

Let A and B are two fixed points on the curve $y=f(x)$ whose abscissae are respectively a and b. Let $P(x, y)$ be any variable point on the curve at an arc distance S from A. Let the area of the surface generated by revolving the arc AP about x axis be σ. Then σ is a function of x.



Let $Q(x+\delta x, y+\delta y)$ be a point on the curve very near to P, so that arc $PQ=\delta S$ and the surface area generated by revolving the arc PQ about x axis is $\delta\sigma$.

Now, by revolving the chord PQ about x axis a frustum of Cone with slant height PQ is obtained.

The area of the surface of this frusturm

$$=\pi(PL+QM)PQ$$

$$=\pi(y+y+\delta y).PQ$$
$$=\pi(2y+\delta y)PQ=\delta\Gamma \text{ (say).}$$

Now $\quad Lt\ \dfrac{\delta\Gamma}{\delta x}=Lt\ \dfrac{\delta\sigma}{\delta x}$

$$\Rightarrow \frac{d\Gamma}{dx}=\frac{d\sigma}{dx}. \qquad (1)$$

But $\dfrac{\delta\Gamma}{\delta x}=\pi(2y+\delta y)\dfrac{PQ}{\delta x}$

$$=\pi(2y+\delta y)\frac{\text{chord } PQ}{\text{arc } PQ}\cdot\frac{\text{arc } PQ}{\delta x}.$$

$\therefore$ In the limit, when $Q\to P$ i.e., $\delta x\to 0$

$$\frac{d\Gamma}{dx}=2\pi y\,.\,1\,.\frac{ds}{dx}$$

So, from (1), we get

$$\frac{d\sigma}{dx}=2\pi y\,\frac{ds}{dx}.$$

Hence, integrating between the limits a and b, we get the surface area obtained by revolving the arc AB about x axis,

$$=2\pi\int_a^b y\,\frac{ds}{dx}\,dx$$

$$=2\pi\int_a^b y\,\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Cor. In case of polar curve $r=f(\theta)$ where θ is the independent variable, the surface area

$$=2\pi\int y\,\frac{ds}{d\theta}\,d\theta.$$

In case of parametric curve $x=f(t)$, $y=\phi(t)$ where t is the parameter, surface area

$$=2\pi\int y\,\frac{ds}{dt}\,.\,dt.$$

Illustrations :

Ex. 1. The circle $x^2+y^2=a^2$ revolves about the x axis. Find the surface area of the whole volume so generated.

The parametric equation of the circle is

$$x=a\cos\theta,\ y=a\sin\theta.$$

Now, the whole surface is generated by a complete revolution of the upper half of the circle.

$$\therefore\quad S=2\pi\int_0^\pi y\frac{ds}{d\theta}\,d\theta$$

$$=2\pi\int_0^\pi a\sin\theta\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$=2\pi\int_0^\pi a\sin\theta\sqrt{a^2\sin^2\theta+a^2\cos^2\theta}\,d\theta$$

$$=2\pi a^2\int_0^\pi \sin\theta\,d\theta=2\pi a^2\Big[-\cos\theta\Big]_0^\pi$$

$$=4\pi a^2.$$

Ex. 2. Find the surface area of the volume formed by a complete revolution of the ellipse $x=a\cos\phi$, $y=b\sin\phi$ about the minor axis ($a>b$).

The complete surface is formed by a complete revolution of the right half of the ellipse about the minor axis between the limits $y=-b$ to b i.e., for $\phi=-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

$$\therefore\quad S=2\pi\int_{-\pi/2}^{\pi/2} x\frac{ds}{d\phi}\,d\phi.$$

$$=2\pi\int_{-\pi/2}^{\pi/2} a\cos\phi\sqrt{\left(\frac{dx}{d\phi}\right)^2+\left(\frac{dy}{d\phi}\right)^2}\,d\phi$$

$$=2\pi\int_{-\pi/2}^{\pi/2} a\cos\phi\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}\,d\phi$$

$$=2\pi\int_{-\pi/2}^{\pi/2} a\cos\phi\sqrt{a^2\sin^2\phi+a^2(1-e^2)\cos^2\phi}\,d\phi$$

$$=2\pi a^2\int_{-\pi/2}^{\pi/2}\cos\phi\,(1-e^2+e^2\sin^2\phi)^{\frac{1}{2}}\,d\phi.$$

$$=4\pi a^2\int_0^{\pi/2}\cos\phi\,(1-e^2+e^2\sin^2\phi)^{\frac{1}{2}}\,d\phi.$$

[Put $e\sin\phi=z$, $e\cos\phi\,d\phi=dz$]

$$=\frac{4\pi a^2}{e}\int_0^e\{z^2+(1-e^2)\}^{\frac{1}{2}}\,dz$$

$$=\frac{4\pi a^2}{e}\left[\frac{z(z^2+1-e^2)}{2}\right.$$

$$\left.+\frac{1-e^2}{2}\log\left\{z+\sqrt{z^2+1-e^2}\right\}\right]_0^e$$

$$=\frac{2\pi a^2}{e}\left[e+(1-e^2)\log(e+1)-(1-e^2)\log\sqrt{1-e^2}\right]$$

$$=2\pi a^2\left[1+\frac{1-e^2}{e}\log\frac{e+1}{\sqrt{1-e^2}}\right]$$

$$=2\pi a^2\left[1+\frac{1-e^2}{e}\log\sqrt{\frac{1+e}{1-e}}\right].$$

Ex. 8. Find the surface area of the volume formed by the revolution of the ellipse $x=a\cos\phi$, $y=b\sin\phi$ about the major axis.

The required surface is formed by the revolution of the upper half of the ellipse about the x axis between the limits $x=a$ to $-a$ i.e, for $\phi=0$ to π.

$$\therefore \quad S=2\pi\int_0^\pi y\,\frac{ds}{d\phi}\,d\phi$$

$$=2\pi\int_0^\pi b\sin\phi\sqrt{\left(\frac{dx}{d\phi}\right)^2+\left(\frac{dy}{d\phi}\right)^2}\,d\phi$$

$$=2\pi\int_0^\pi b\sin\phi\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}\,d\phi$$

$$=2\pi\int_0^\pi b\sin\phi\sqrt{a^2\sin^2\phi+a^2(1-e^2)\cos^2\phi}\,d\phi$$

$$[\because\ b^2=a^2(1-e^2)]$$

$$=2\pi ab\int_0^\pi \sin\phi\,(1-e^2\cos^2\phi)^{\frac{1}{2}}\,d\phi$$

[Put $e\cos\phi=\sin\theta$, then $-e\sin\phi\,d\phi=\cos\theta d\theta$.]

$$=2\pi ab\int_{\sin^{-1}e}^{-\sin^{-1}e} -(1-\sin^2\theta)^{\frac{1}{2}}\,\frac{1}{e}\cos\theta\,d\theta$$

$$=\frac{2\pi ab}{e}\int_{-\sin^{-1}e}^{\sin^{-1}e}\cos^2\theta\,d\theta$$

$$=\frac{4\pi ab}{e}\int_0^{\sin^{-1}e}\cos^2\theta\,d\theta \quad [\because\ \cos^2\theta \text{ is an even function of } \theta.]$$

$$=\frac{2\pi ab}{e}\int_0^{\sin^{-1}e}(1+\cos 2\theta)\,d\theta=\frac{2\pi ab}{e}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\sin^{-1}e}$$

$$=\frac{2\pi ab}{e}\left[\sin^{-1}e+e\sqrt{1-e^2}\right]$$

$$=2\pi ab\left\{\frac{1}{e}\sin^{-1}e+\sqrt{1-e^2}\right\}.$$

Ex. 4. The arc of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ from $x=a$ to 0 revolves about x axis. Find the surface area so form'd. **(C. H. 1968)**

The parametric equation is $x=a\cos^3\phi$, $y=a\sin^3\phi$.

$\therefore$ When $x=a$, $\phi=0$ and when $x=0$, $\phi=\pi/2$

$$\therefore \quad S=2\pi\int_0^{\pi/2} y\frac{ds}{d\phi}d\phi$$

$$=2\pi\int_0^{\pi/2} a\sin^3\phi\sqrt{\left(\frac{dx}{d\phi}\right)^2+\left(\frac{dy}{d\phi}\right)^2}\,d\phi$$

$$=2\pi\int_0^{\pi/2} a\sin^3\phi\sqrt{\{3a\cos^2\phi(-\sin\phi)\}^2+\{3a\sin^2\phi\cos\phi\}^2}\,d\phi$$

$$=2\pi a^2\int_0^{\pi/2}\sin^3\phi\sqrt{9\cos^4\phi\sin^2\phi+9\sin^4\phi\cos^2\phi}\,d\phi$$

$$=6\pi a^2\int_0^{\pi/2}\sin^4\phi\cos\phi\,d\phi$$

$$=6\pi a^2\left[\tfrac{1}{5}\sin^5\phi\right]_0^{\pi/2}$$

$$=\tfrac{6}{5}\pi a^2.$$

Note : The surface of the solid generated by the revolution of the astroid $x=a\cos^3 t$, $y=a\sin^3 t$ about the x axis is

$$S=2.2\pi\int_0^{\pi/2} y\frac{ds}{d\phi}\,d\phi$$

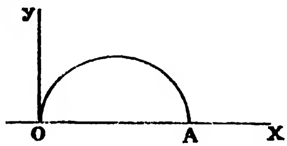
$=2.\frac{6}{5}\pi a^2$ as in *Ex* 4.

$=\frac{12}{5}\pi a^2.$

Ex. 5. Find the surface area of the solid generated by revolving the cycloid

$$\mathbf{x=a(\theta-\sin\theta),\ y=a(1-\cos\theta)}$$

about the x axis.



The extreme values of x for one arc of the cycloid are 0 and $2\pi a$.

The extreme values of θ are 0 and 2π.

$$\therefore\quad S=2\pi\int_0^{2\pi} y\frac{ds}{d\theta}\,d\theta.$$

$$=2\pi\int_0^{2\pi} a(1-\cos\theta)\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$=2\pi a\int_0^{2\pi}(1-\cos\theta)\sqrt{a^2(1-\cos\theta)^2+a^2\sin^2\theta}\,d\theta$$

$$=2\pi a^2\int_0^{2\pi} 2\sin^2\frac{\theta}{2}.\sqrt{2-2\cos\theta}\,d\theta.$$

$$=8\pi a^2\int_0^{2\pi}\sin^3\frac{\theta}{2}\,d\theta \qquad \left[\text{Put } \frac{\theta}{2}=z,\ d\theta=2dz\right]$$

$$=16\pi a^2 \int_0^{\pi} \sin^3 z \quad dz$$

$$=32\pi a^2 \int_0^{\pi/2} \sin^3 z \quad dz.$$

$$=32\pi a^2 . \tfrac{2}{3} . 1$$

$$=\tfrac{64}{3}\pi a^2 .$$

Ex. 6. Find the surface area of the solid formed by revolving cardioide

$$\mathbf{r=a(1+\cos\theta)}$$

about the initial line.

Here, $S=2\pi \int_0^{\pi} y\dfrac{ds}{d\theta}\, d\theta \qquad \cdots \qquad \cdots \qquad (1)$

Now $\quad y=r\sin\theta=a(1+\cos\theta)\sin\theta$

and $\quad \dfrac{ds}{d\theta}=\sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}$

$$=\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}$$

$$=2a\cos\frac{\theta}{2}.$$

$\therefore$ From (1)

$$S=2\pi\int_0^{\pi} a(1+\cos\theta)\sin\theta \, . \, 2a\cos\frac{\theta}{2}\, d\theta$$

$$=4\pi a^2\int_0^{\pi} 2\cos^2\frac{\theta}{2} \, . \, 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \, . \, \cos\frac{\theta}{2}\, d\theta$$

$$=16\pi a^2\int_0^{\pi} \sin\frac{\theta}{2}\cos^4\frac{\theta}{2}\, d\theta \qquad \left[\text{Put } \frac{\theta}{2}=z,\ d\theta=2dz\right]$$

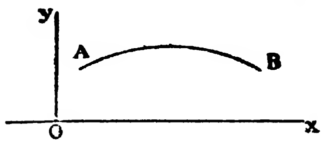
$$=32\pi a^2\int_0^{\pi/2}\sin z\cos^4 z\,dz.$$

$$=32\pi a^2\left[-\tfrac{1}{5}\cos^5 z\right]_0^{\pi/2}$$

$$=\tfrac{32}{5}\pi a^2.$$

16.6. Surface of revolution : Theorem

The area of the surface of revolution generated by an arc which does not meet the axis of revolution is equal to the product of the length of the arc by the circumference described by the centre of gravity of the arc.



Let l be the length of the curve AB measured from A.

Then the surface of the solid of revolution about the x axis is

$$S=2\pi\int_0^l y\,ds \text{ where } y \text{ is a function of } S. \qquad (1)$$

Let $\bar{y}$ be the distance of the centroid of the arc from the x axis

Then $\bar{y}=\dfrac{\int_0^l y\,ds}{l}$.

$$\therefore \int_0^l y\,ds=l\bar{y}.$$

Hence, from (1)

$$S=2\pi l\bar{y}$$

$$=2\pi\bar{y}\times l$$

$=$ **length of path described by the centroid $\times$ length of the curve.**

Illustrations :

Ex. 1. Find the surface of the solid formed by revolving the ellipse $x^2/a^2+y^2/b^2=1$ about the line $x=2a$, $a>b$.

Origin being the centre of the ellipse, the distance of the *C.G.* from the line $x=2a$ is clearly $2a$.

$\therefore$ The circumference described by $2a$ is $2\pi(2a)$ i.e, $4\pi a$.

Also, perimeter of the ellipse is

$$2\pi a\left\{1-\left(\frac{1}{2}\right)^2\frac{e^2}{1}-\left(\frac{1.3}{2.4}\right)^2\frac{e^4}{3}\cdots\cdots\right\}$$

Hence, required surface area

$$=4\pi a\,.\,2\pi a\left\{1-\left(\frac{1}{2}\right)^2\frac{e^2}{2}-\left(\frac{1.3}{2.4}\right)\frac{e^4}{3}\cdots\cdots\right\}$$

$$=8\pi^2a^2\left\{1\quad\left(\frac{1}{2}\right)^2\frac{e^2}{2}-\left(\frac{1.3}{2.4}\right)\frac{e^4}{3}-\cdots\cdots\right\}$$

Ex. 2. Find the surface of the solid formed by the revolution of the evolute of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

about the line $x=a$.

The centre of the evolute is the origin. So, the distance of C.G. from the line $x=a$ is a.

$\therefore$ The circumference described by a length$=a$ is $2\pi a$.

But the entire length of the evolute of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$=4\left(\frac{a^2}{b}-\frac{b^2}{a}\right).$$

Hence, the required surface area

$$=2\pi a\,.\,4\left(\frac{a^2}{b}-\frac{b^2}{a}\right)$$

$$=8\pi a\left(\frac{a^2}{b}-\frac{b^2}{a}\right).$$

Exercise 16

1. Show that the volume of a right circular cone whose base is of radius a and height h is $\frac{1}{3}\pi a^2 h$.

2. Show that the volume of the parabola $\sqrt{x}+\sqrt{y}=\sqrt{a}$ generated by the revolution about the x axis between the limits $x=0$, $y=0$ is $\frac{1}{15}\pi a^3$.

3. The arc of astroid $x^{2/3}+y^{2/3}=a^{2/3}$ from $\theta=0$ to $\theta=\pi/2$ revolves about the x axis. Show that it generates a volume $\frac{16}{105}\pi a^3$ of surface area $\frac{6}{5}\pi a^2$.

4. Show that the volume of the solid formed by revolving the circle $x^2+y^2=a^2$ about the line $x=2a$ is $4\pi^2 a^3$.

5. If the curve included between the two cusps of the cycloid $x=a(\theta+\sin\theta)$, $y=a(1+\cos\theta)$ is revolved about the x axis, then the area of the surface so generated bears to the area of the cycloid a ratio 64 : 9.

6. The part of the parabola $y^2=4ax$ bounded by the latus rectum revolves about the y axis. Show that the surface so generated is

$$\pi a^2\{3\sqrt{2}-\log(\sqrt{2}+1)\}.$$

7. Show that the volume and surface generated by the revolution of the cycloid

$$x=a(\theta+\sin\theta),\ y=a(1+\cos\theta)$$

about the base are in the ratio 15π : 64.

8. Show that the volume and surface generated by the revolution of the cardioide $r=a(1-\cos\theta)$ about the initial line are in the ratio $5a$: 12.

9. Show that the surface area of the solid generated by the revolution of the lemniscate $r^2=a^2\cos 2\theta$ about the initial line is $2\sqrt{2}\pi a^2(\sqrt{2}-1)$.

(B. H. 1971)

10. Show that the volume generated by the revolution of one loop of the lemniscate about the line $\theta=\pi/2$ is $\pi^2 a^3/4\sqrt{2}$.

11. The area enclosed by the parabola $x^2=4ay$ and $x^2=4a(2a-y)$ revolves about the line $y=a$. Show that the volume so generatad is $32\pi a^3/15$.

(B. H. 1970)

12. The curve $y=\dfrac{b}{\sqrt{\pi}}e^{-b^2x^2}$ revolves about the line $x=0$. Show that the volume so generated is $\sqrt{\pi}/b$.

13. Show that the surface area formed by the revolution of the curve $r=2a\cos\theta$ about the initial line is $4\pi a^2$.

14. The area bounded by a quadrant of the circle $x^2+y^2=a^2$ revolves about x or y axis. Show that the volume so generated is

$$(10-3\pi)\pi a^3/6.$$

15. Show that the volume and the surface of the solid generated by the revolution of the loop of the curve $x=t^2$ and $y=t-\frac{1}{3}t^3$ are in the ratio of 1 : 4.

16. The ellipse $x=a\cos\theta$, $y=b\sin\theta$ $(a>b)$ revolves about the tangent at one extremity of its minor axis.. Show that the volume so generated is $2\pi^2ab^2$.

17. Show that surface area of the frustum of a cone is $\pi\times$slant height $\times$sum of the radii of two bases.

18. Show that the volume generated by the revolution of the curve

$$y^2=\frac{a^2}{a-x}x$$

about its asymptote is $\frac{1}{2}\pi^2a^3$.

PART-III

APPLICATION OF MECHANICS

CHAPTER XVII

CENTRE OF GRAVITY

17.1. Centre of gravity is often called as centre of mass or centre of Inertia.

Suppose n particles of weights $w_1, w_2, \cdots w_r, \cdots w_n$, are distributed on a plane at points with co-ordinates

$$(x_1, y_1), (x_2, y_2), \cdots\cdots (x_r, y_r), \cdots\cdots (x_n, y_n).$$

If $(\bar{x}, \bar{y})$ be the co-ordinate of the centre of gravity (C.G.) of the system lying on the plane, then by statics we know

$$\bar{x} = \frac{\sum_{r=1}^{n} w_r x_r}{\sum_{r=1}^{n} w_r}, \quad \bar{y} = \frac{\sum_{r=1}^{n} w_r y_r}{\sum_{r=1}^{n} w_r}.$$

Since, $w_r = m_r g$ the same thing may be written after cancelling g as

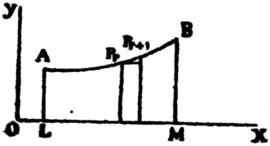
$$\bar{x} = \frac{\sum_{r=1}^{n} m_r x_r}{\sum_{r=1}^{n} m_r}, \quad \bar{y} = \frac{\sum_{r=1}^{n} m_r y_r}{\sum_{r=1}^{n} m_r}.$$

Because of this relation, the centre of gravity is often called the centre of mass or centre of inertia.

17.2. Centre of gravity of a uniform plane curve.

Let $y = f(x)$ be a continuous curve intercepted between two points A and B where abscissae are a and b i.e. $OL = a$ and $OM = b$.

Let us divide the interval (a, b) by points



$a=x_0, x_1, x_2, \cdots\cdots x_r \cdots x_n=b$

such that $x_r=a+rh$.

Let the ordinate at these points meets the curve at

$$A=P_0, P_1, P_2, \cdots\cdots P_{r-1}, P_r, \cdots\cdots P_n=B.$$

and let arc $AP_r=S_r$.

If ρ be the mass per unit length of the curve then the mass of the arc $P_rP_{r+1}=\rho(S_{r+1}-S_r)$.

Since, this arc is very small, this mass may be assumed to be placed at (x_r, y_r)

With this assumption, we can say that n particles with masses $\rho(S_1-S_0)$, $\rho(S_2-S_1)\cdots\rho(S_{r+1}-S_r)\cdots\rho(S_n-S_{n-1})$ are distributed at n points with co-ordinates

$$(x_0, y_0), (x_1, y_1)\cdots(x_r, y_r)\cdots(x_{n-1}, y_{r-1}).$$

If $(\bar{x}, \bar{y})$ be the C.G. of the system then

$$\bar{x}=\frac{\sum_{r=0}^{n-1}\rho(S_{r+1}-S_r)x_r}{\sum_{r=0}^{n-1}\rho(S_{r+1}-S_r)}=\frac{\sum_{r=0}^{n-1}(S_{r+1}-S_r)\,x_r}{\sum_{r=0}^{n-1}(S_{r+1}-S_r)}$$

$$=\frac{1}{l}\sum_{r=0}^{n-1}(S_{r+1}-S_r)\,x_r$$

$$\bar{y}=\frac{\sum_{r=1}^{n-1}\rho(S_{r+1}-S_r)y_r}{\sum_{r=0}^{n-1}\rho(S_{r+1}-S_r)}=\frac{\sum_{r=0}^{n-1}(S_{r+1}-Sr)\,y_r}{\sum_{r=0}^{n-1}(S_{r+1}-S_r)}$$

$$=\frac{1}{l}\sum_{r=0}^{n-1}(S_{r+1}-S_r)\,y_r \qquad \dots \quad (1)$$

$$\because \; l = \text{length of the curve} = \sum_{r=1}^{n-1}(S_{r+1}-S_r)$$

Now, $\underset{x_{r+1}\to x_r}{Lt}(S_{r+1}-S_r)\,x_r$

$$= \underset{x_{r+1}\to x_r}{Lt}\, x_r \frac{S_{r+1}-S_r}{x_{r+1}-x_r}(x_{r+1}-x_r)$$

$$= x_r.h \underset{x_{r+1}\to x_r}{Lt}\frac{S_{r+1}-S_r}{x_{r+1}-x_r}$$

$$=\left(\frac{dS}{dx}\right)_{x=x_r}.$$

So, from (1), we can write

$$\bar{x}=\frac{h}{l}\sum_{r=0}^{n-1}x_r\left(\frac{dS}{dx}\right)_{x=x_r} \qquad \dots \quad (2)$$

$$\bar{y}=\frac{h}{l}\sum_{r=0}^{n-1}y_r\left(\frac{dS}{dy}\right)_{y=y_r} \qquad \dots \quad (3)$$

Now, writing $x\dfrac{dS}{dx}=\phi(x)$ and $y\dfrac{dS}{dy}=\psi(x)$

we get,

$$h\sum_{r=0}^{n-1}x_r\left(\frac{dS}{dx}\right)_r=h\sum_{r=0}^{n-1}\phi(a+rh)$$

$$=\int_a^b \phi(x)\,dx$$

[In the limit as $h\to 0$, by fundamental Theorem of Integral Calculus.]

$$=\int_a^b x\frac{dS}{dx}dx$$

Hence, from (2) $\bar{x}=\frac{1}{l}\int_a^b x\frac{dS}{dx}.\ dx.$

Similarly from (2), $\bar{y}=\frac{1}{l}\int_a^b y\frac{dS}{dy}.\ dy.$

Ex. Find the centre of gravity of the arc lying on the first quadrant of the astroid

$$\mathbf{x=a\cos^3\theta,\ y=a\sin^3\theta.}$$

In the first quadrant, the arc of the astroid extends from 0 to a

$$\therefore\quad \bar{x}=\frac{\int_0^a x\frac{dS}{dx}\,dx}{\int_0^a \frac{dS}{dx}\,dx}$$

$$=\frac{\int_0^{\pi/2} x\frac{dS}{dx}\cdot\frac{dx}{d\theta}\cdot d\theta}{\int_0^{\pi/2}\frac{dS}{dx}\cdot\frac{dx}{d\theta}\,d\theta}\quad\frac{\int_0^{\pi/2} x\frac{dS}{d\theta}\,d\theta}{\int^{\pi/2}\frac{dS}{d\theta}\,d\theta.}$$

$$=\frac{\int_0^{\pi/2} x\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta}{\int_0 \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)}\,d\theta}$$

$$=\frac{\int_0^{\pi/2} a\cos^3\theta\,\sqrt{9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta}\,d\theta}{\int_0^{\pi/2}\sqrt{9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta}\,d\theta}$$

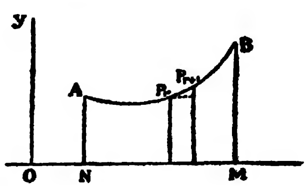
$$=\frac{\int_0^{\pi/2} a\cos^3\theta\,.\,3a\sin\theta\cos\theta\,d\theta}{\int_0^{\pi/2} 3a\sin\theta\cos\theta\,d\theta}$$

$$=\frac{3a^2\int_0^{\pi/2}\sin\theta\cos^4\theta\,d\theta}{\frac{3a}{2}\int_0^{\pi/2}\sin 2\theta\,d\theta}$$

$$=\frac{3a^2\left[-\frac{1}{5}\cos^5\theta\right]_0^{\pi/2}}{\frac{3a}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}}=\frac{\frac{3a^2}{5}}{\frac{3a}{2}}=\frac{2a}{5}.$$

Similarly, or from symmetry $\bar{y}=\frac{2a}{5}$.

17.3. Centre of Gravity of a uniform plane area.



Let us consider the plane area bounded by the curve $y=f(x)$, the x axis and the ordinates at $x=a$ and $x=b$.

Let the interval (a, b) be divided into n equal parts each part being $=h$ by the points

$$a=x_0, x_1, x_2 \cdots\cdots x_r, x_{r+1}, \cdots\cdots x_n=b$$

so that $x_r=a+rh.$

Let the ordinates at these points meet the curve at

$$A=P_0, P_1, P_2, \cdots\cdots P_r, P_{r+1}, \cdots\cdots P_n=B$$

so that ordinate through $P_r=y_r$

Then $y_r=f(x_r)=f(a+rh)$

Since, the arc $P_r P_{r+1}$ is very small, the area under this curve is nearly $=hy_r$

$\therefore$ Its mass $=\rho hy_r.$

The $C.G.$ of this area is at $\{\frac{1}{2}(x_r+x_{r+1}), \frac{1}{2}y_r\}$

Replacing the rectangle by a particle of mass ρhy_r at its $C.G.$ the $C.G.$ of the n particles are given by

$$\bar{x}=\frac{\Sigma\rho hy_r.\frac{1}{2}(x_r+x_{r+1})}{\Sigma\rho hy_r} \qquad \cdots \qquad \cdots \qquad (1)$$

$$\bar{y}=\frac{\Sigma\rho hy_r.\frac{1}{2}y_r}{\Sigma\rho hy_r} \qquad \cdots \qquad \cdots \qquad \cdots \qquad (2)$$

Now, proceeding to the limits as $h\to 0$ i.e., $n\to\infty$

$$Lt \quad \Sigma hy_r.\tfrac{1}{2}(x_r+x_{r+1})=\int_a^b yx\,dx$$

$$Lt \quad \Sigma hy_r=\int_a^b y\,dx.$$

and $Lt \quad \Sigma\frac{1}{2}hy_r^2=\frac{1}{2}\int_a^b y^2dx.$

Hence, from (1) and (2)

$$\bar{x}=\frac{\int_a^b xy\,dx}{\int_a^b y\,dx} \quad \text{and} \quad \bar{y}=\frac{\frac{1}{2}\int_a^b y^2dx}{\int_a^b y\,dx}.$$

Cor. The C.G. of the plane area bounded by the curve $x=f(y)$, the y axis and the ordinates $y=c$ and $y=d$ can be similarly shown to be

$$\bar{x}=\frac{\frac{1}{2}\int_c^d x^2 dy}{\int_c^d x\,dy}, \quad \bar{y}=\frac{\int_c^d yx\,dx}{\int_c^d x\,dy}.$$

Ex. Find the C.G. of a uniform lamina bounded by the arc of the ellipse $x^2/a^2+y^2/b^2=1$ and the two axes of coordinates. [C. H. 1965 (old)]

The parametric equation is $x=a\cos\theta$, $y=b\sin\theta$.

$$\bar{x}=\frac{\int_0^a xy\,dx}{\int_0^a y\,dx}=\frac{\int_{\pi/2}^0 a\cos\theta\,.\,b\sin\theta(-a\sin\theta)d\theta}{\int_{\pi/2}^0 b\sin\theta(-a\sin\theta)d\theta}$$

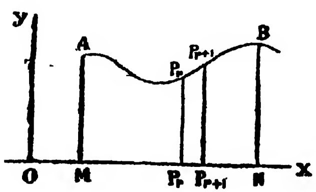
$$=a\frac{\int_0^{\pi/2}\cos\theta\sin^2\theta\,d\theta}{\int_0^{\pi/2}\sin^2\theta\,d\theta}=a\frac{\left[\frac{1}{3}\sin^3\theta\right]_0^{\pi/2}}{\frac{\pi}{4}}=\frac{4a}{3\pi}$$

$$\bar{y}=\frac{\frac{1}{2}\int_0^a y^2dx}{\int_0^a y\,dx}=\frac{\frac{1}{2}\int_{\pi/2}^0 b^2\sin^2\theta(-a\sin\theta)d\theta}{\int_{\pi/2}^0 b\sin\theta(-a\sin}$$

$$=\frac{b}{2}\frac{\int_0^{\pi/2}\sin^3\theta\, d\theta}{\int_0^{\pi/2}\sin^2\theta\, d\theta}=\frac{b}{2}\frac{\frac{2}{3}}{\frac{\pi}{4}}=\frac{4b}{3\pi}.$$

Hence, the required $C.G.$ is at $\left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right)$.

17.4. Centre of gravity of a volume of revolution.



Let the curve $y=f(x)$ intercepted between two points with abscissae a and b revolves about the x axis. Let the ordinates through A and B meet the x axis at M and N.

Then $OM=a$ and $ON=b$.

Divide the interval (a, b) by the points

$$a=x_0,\ x_1,\ x_2,\ \cdots\cdots x_r,\ x_{r+1},\cdots\cdots x_n=b$$

so that $x_r=a+rh$.

Let the ordinates at these points meet the curve at $A=P_0,\ P_1,\ \ P_2\cdots P_r,\ P_{r+1}\cdots P_n=B$ and let the ordinate through P_r be y_r.

Then $y_r=f(a+rh)$.

Now, the volume obtained by revolving the rectangle P_rP_{r+1} about the x axis

$$=\rho(\pi y_r^2).h.$$

Its $C.G.$ is at a point on the x axis whose abscissa $=\frac{1}{2}(x_r+x_{r+1})$.

Let us now replace this volume by a particle of each mass at a point whose abscissa $=\frac{1}{2}(x_r+x_{r+1})$ and $(\bar{x}, \bar{y})$ be the $C.G.$ of the system of n particles. Then

$$\bar{x}=\frac{\sum_{r=0}^{n-1}\rho\pi y_r^2.h.\tfrac{1}{2}(x_r+x_{r+1})}{\sum_{r=0}^{n-1}\rho\pi y_r^2.h}=\frac{\sum_{r=0}^{n-1}\tfrac{1}{2}(x_r+x_{r+1})y_r^2}{\sum_{r=0}^{n-1}y_r^2}$$

$$=\frac{\int_a^b xy^2dx}{\int_a^b y^2\,dx}\qquad \because \text{ In the limits } n\to\infty \text{ as } h\to 0$$

Also $\bar{y}=0$. Hence, $C.G.$ of the volume of revolution are at

$$\left(\frac{\int_a^b xy^2dx}{\int_a^b y^2dx},\ 0\right)$$

Cor. As in the last article the $C.G.$ of the surface of revolution are given by

$$\bar{x}=\frac{\int_a^b xy\,\frac{ds}{dx}\,dx}{\int_a^b y\,\frac{ds}{dx}\,dx}\quad\text{and}\quad \bar{y}=0.$$

Ex. Find the C.G. of a solid hemisphere.

The solid hemisphere is generated by the revolution of one quadrant of the circle $x^2+y^2=a^2$ about any one of the two axes.

Considering the revolution of the area in the first quadrant about x axis,

$$\bar{x}=\frac{\int_0^a xy^2dx}{\int_0^a y^2dx}=\frac{\int_0^a x(a^2-x^2)dx}{\int_0^a (a^2-x^2)dx}$$

$$=\frac{\left[a^2\frac{x^2}{2}-\frac{x^4}{4}\right]_0^a}{\left[a^2x-\frac{x^3}{3}\right]_0^a}=\frac{\frac{a^4}{2}-\frac{a^4}{4}}{a^3-\frac{a^3}{3}}=\frac{\frac{1}{4}a^4}{\frac{2}{3}a^3}=\frac{3}{8}a.$$

Exercise 17

1. Find the centroid of the area bounded by $y^2(2a-x)=x^3$ and its asymphote. Ans. $\bar{x}=\frac{5}{3}a,\ \bar{y}=0$

2. Show that the centroid of the arc of the catenary $y=\frac{1}{2}(e^x+e^{-x})$ lying between $x=-1$ and $x=1$ is $\bar{y}=\frac{e^2+4-e^{-2}}{4(e-e^{-1})}$.

3. Where is the centroid of the triangle whose vertices are $(0, 0)$, (x_1, y_1) and (x_2, y_2) ? Ans. $\bar{x}=\frac{1}{3}(x_1+x_2)$, $\bar{y}=\frac{1}{3}(y_1+y_2)$

4. Show that $C.G.$ of the sector of a circle of radius a is on the radius bisecting the sector at a distance $\frac{2}{3}a\ \frac{\sin\alpha}{\alpha}$ from the centre, 2α being the angle of the sector at the centre.

5. Show that $C.G.$ of a uniform lamina bounded by the parabola $y^2=4ax$ and a double ordinate at $x=x_1$ divides x_1 in the ratio 3 : 2.

6. Show that $C.G.$ of the area of the parabola $y^2=4ax$ intercepted between the vertex and one extremity of the latus-rectum have co-ordinates

$$\bar{x}=\frac{a}{4}\,\frac{3\sqrt{2}-\log(\sqrt{2}+1)}{\sqrt{2}+\log(\sqrt{2}+1)},\quad \bar{y}=\frac{4a}{3}\,\frac{2\sqrt{2}-1}{\sqrt{2}+\log(\sqrt{2}+1)}.$$

7. Show that the centroid of the whole arc of the cardioide $r=a(1+\cos\theta)$ lie on the initial line at a distance $\frac{4}{5}a$ from the pole.

8. Show that the centroid of the area bounded by the cardiode $r=a(1+\cos\theta)$ lie on the initial line at a distance $\frac{5}{6}a$ from the pole.

9. Show that the centroid of the area under one arc of the sine curve $y=\sin x$ have co-ordinates $\bar{x}=\pi/2$, $\bar{y}=\pi/8$.

10. Show that the centroid of the surface formed by the revolution of the cardioide $r=a(1+\cos\theta)$ about the initial line is given by $\bar{x}=\frac{50}{63}a$.

CHAPTER—XVIII

MOMENT OF INERTIA

18.1. Moment of Inertia and radius of gyration.

Let $m_1, m_2, m_3, \cdots\cdots$ be the elementary masses of a body at distances $r_1, r_2, r_3 \cdots$ respectively from a given line. Then *moment of inertia* of the body about the line

$$= m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots\cdots$$
$$= \Sigma m r^2$$

If M be the total mass of the body and k be a number such that

$$Mk^2 = \Sigma m r^2$$

then k is called the radius of gyration about the given line.

18.2. Special Cases of Moments of Inertia.

(1) Moment of Inertia of a thin uniform rod.

Let PQ be any element of the rod AB such that

$$AP = x \text{ and } PQ = \delta x.$$

If $2a$ be the length of the rod and M be its mass then,

the mass of PQ is $\dfrac{\delta x}{2a} M$.

Hence, the moment of inertia of the thin rod about a line through A perpendicular to the rod

$$= \sum \frac{\delta x}{2a} M x^2$$

$$= \frac{M}{2a} \int_0^{2a} x^2 \, dx$$

$$= \frac{M}{2a} \left\{ \frac{x^3}{3} \right\}_0^{2a} = \frac{M}{2a} \cdot \frac{8a^3}{3} = M \cdot \frac{4a^2}{3}.$$

Again if O be the mid point of the rod such that $OP=x$ and $PQ=\delta x$, then the moment of inertia of the rod about an axis through the mid point perpendicular to the rod

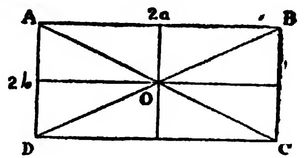
$$=\sum \frac{\delta x}{2a} M.x^2$$

$$=\frac{M}{2a}\int_{-a}^{a} x^2\ dx=\frac{M}{2a}\left[\frac{x^3}{3}\right]_{-a}^{a}$$

$$=\frac{M}{6a}\cdot 2a^3=M\cdot\frac{a^2}{3}.$$

(2) Moment of Inertia of a Rectangular lamina.

Let $ABCD$ be a rectangular lamina of sides $2a$ and $2b$.



The lamina may be divided by strips parallel to AD.

The moment of inertia of each of these strips about an axis through O, the mid point of the lamina, parallel to AB=mass of the strip $\times \frac{b^2}{3}$.

So, the moment of inertia of the rectangle about the line through O parallel to AB

=Sum of the moments of inertia of all the strips

$$=M\cdot\frac{b^2}{3}.$$

Similarly, the moment of inertia of the rectangle about a line through O parallel to AD

$$=M\frac{a^2}{3}.$$

Let the two lines through O parallel to AB and AD be the axis of x and y respectively and $P(x, y)$ be any point of the lamina.

Then, the moment of inertia about $OX=\Sigma my^2$ and about $OY=\Sigma mx^2$.

So, we get,

$$\Sigma my^2=M\frac{b^2}{3} \text{ and } \Sigma mx^2=M\frac{a^2}{3}.$$

Hence, the moment of inertia of the lamina

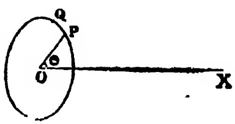
$$=\Sigma m.OP^2$$

$$=\Sigma m(x^2+y^2)=\Sigma mx^2+\Sigma my^2$$

$$=M\frac{a^2}{3}+\frac{Mb^2}{3}$$

$$=M.\frac{a^2+b^2}{3}$$

(3) **Moment of Inertia of the circumference of a circle.**



Let OX be an axis through O, the centre of the circle on the plane of the circle.

Let PQ be an element of the circumference such that

$$\angle POX=\theta \text{ and } \angle POQ=\delta\theta.$$

Then, arc $PQ=a.\delta\theta$ if a be the radius of the circle and the mass of this element

$$=\frac{a\delta\theta}{2\pi a}\text{ M.}$$

Also, the perpendicular distance of P from OX

$$=OP\sin\theta=a\sin\theta.$$

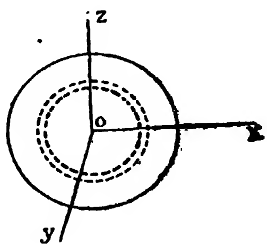
$\therefore$ the moment of inertia of circumference about OX

$$=\sum\left(\frac{a\,\delta\theta}{2\pi a}M\right)a^2\sin^2\theta$$

$$=\frac{Ma^2}{2\pi}\int^{2\pi} \sin^2\theta\, d\theta$$

$$=4.\frac{Ma^2}{2\pi}\int^{\pi/2} \sin^2\theta\ d\theta=\frac{2Ma^2}{\pi}.\frac{\pi}{4}=M.\frac{a^2}{2}.$$

(4) Moment of Inertia of a circular disc of radius a about an axis through the centre perpendicular to the disc.



Consider two concentric circles of radius r and $r+\delta r$.

Then the area contained between these to circles

$$=2\pi r.\delta r$$

$\therefore$ Its mass $=\frac{2\pi r\ \delta r}{\pi a^2}\ M$

So, its moment of inertia about a diameter

$$=\frac{2r\ \delta r}{a^2}\ M\ .\frac{r^2}{2} \quad \text{(by the article No. 3)}$$

$\therefore$ The moment of inertia of the disc about a diameter

$$=\frac{M}{a^2}\int_0^a r^3\, dr=M\ \frac{a^2}{4}.$$

$\therefore$ The moment of inertia of the disc about a perpendicular diameter is also

$$=M\frac{a^2}{4}.$$

Let these two diameters be the axis of X and Y respectively. If $P(x, y)$ be any point on the disc, then the moment of inertia of the disc about OX and OY are respectively

Σmy^2 and Σmx^2

$$\therefore \quad \Sigma mx^2 = M\frac{a^2}{4}.$$

$$\text{and } \Sigma my^2 = M\frac{a^2}{4}.$$

Hence, the moment of inertia of the disc about an axis through O perpendicular to the disc

$$= \Sigma m\, OP^2$$
$$= \Sigma m(x^2+y^2) = \Sigma mx^2 + \Sigma my^2$$
$$= M\frac{a^2}{4} + M\frac{a^2}{4} = M \cdot \frac{a^2}{2}.$$

(5) **Moment of Inertia of a sphere about a diameter.**

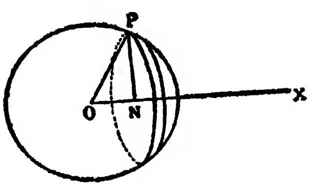
(C. H. 1964 Old)

Let a be the radius and M be the mass of the sphere. Then the volume of the sphere

$$= \frac{4}{3}\pi a^3.$$

$$\therefore \quad \text{Its density} = \frac{3M}{4\pi a^3}.$$

Divide the sphere into infinitely thin circular discs perpendicular to the axis OX about which the moment of inertia is required.



The volume of the disc contained between the planes x and $x+\delta x = \pi\ (a^2 - x^2)\ \delta x.$

$$\because \ PN^2 = a^2 - x^2$$

$\therefore$ Moment of inertia of this disc about OX which is perpendicular to the disc through its centre

$$= \frac{3M}{4\pi a^3} \cdot \pi(a^2 - x^2)\ \delta x \cdot \frac{a^2 - x^2}{2} \quad \text{(by article No. 4)}$$

$\therefore$ Required moment of Inertia

$$=\int_{-a}^{a}\frac{3M}{4\pi a^3}\cdot \pi(a^2-x^2)\,.\left(\frac{a^2-x^2}{2}\right)dx$$

$$=\frac{3M}{8a^3}\int_{-a}^{a}(a^2-x^2)^2\,dx$$

$$=\frac{3M}{8a^3}\int_{-a}^{a}(a^4-2a^2x^2+x^4)\,dx$$

$$=\frac{3M}{8a^3}\left[a^4x-2a^2\frac{x^3}{3}+\frac{x^5}{5}\right]_{-a}^{a}$$

$$=\frac{3M}{8a^3}\left[\left(a^5-\frac{2a^5}{3}+\frac{a^5}{5}\right)-\left(-a^5+\frac{2a^5}{3}-\frac{a^5}{5}\right)\right]$$

$$=\frac{3M}{4a^3}\left(a^5-\frac{2a^5}{3}+\frac{a^5}{5}\right)$$

$$=\frac{2}{5}Ma^2.$$

(6) Moment of inertia of the quadrant of an ellipse about the minor axis, the surface density being uniform.

(C. H. 1963)

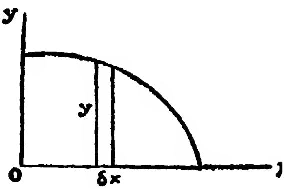
Let the equation of the ellipse be $x^2/a^2+y^2/b^2=1$ and M be the mass of the quadrant.

Then, the area of the quadrant $=\dfrac{\pi ab}{4}$

$\therefore$ Its surface density $=\dfrac{4M}{\pi ab}$.

The area of an elementary strip of breadth $\delta x = y\delta$

$$= b\sqrt{1-\frac{x^2}{a^2}}\,.\,\delta x$$



The moment of inertia of this strip about the minor axis (i.e., y-axis)

$$= \int_0^a \frac{4M}{\pi ab}\,.\, b\sqrt{1-\frac{x^2}{a^2}}\,.\,x^2\,.\,dx$$

$$= \frac{4M}{\pi a^2}\int_0^a x^2\sqrt{a^2-x^2}\,dx$$

[Put $x = a\sin\theta \quad \therefore \quad dx = a\cos\theta\, d\theta$]

$$\therefore \text{ M.I.} = \frac{4M}{\pi a^2}\int_0^{\pi/2} a^2\sin^2\theta\,.\,a^2\cos^2\theta\,d\theta$$

$$= \frac{4Ma^2}{\pi}\,.\,\frac{1}{4}\int_0^{\pi/2}\sin^2 2\theta\,.\,d\theta$$

$$= \frac{Ma^2}{\pi}\,.\,\frac{1}{2}\int_0^{\pi/2}(1-\cos 4\theta)\,d\theta$$

$$= \frac{Ma^2}{2\pi}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{Ma^2}{2\pi}\,.\,\frac{\pi}{2} = M.\frac{a^2}{4}.$$

Exercise 18

1. Find the moment of inertia of a right circular cone, radius of base a about its axis. Ans. $\frac{3}{10}Ma^2$.

2. A bar of 10 units long has a mass of 100 and its density varies directly as the distance from one end. Find the moment of inertia about the lighter end and about the heavier end. Ans. 5000, $1666\frac{2}{3}$

3. Show that the moment of inertia of the volume formed by revolution of the area in the first quadrant under the curve $y^2=x$ between $x=0$ and $x=1$ is $\frac{1}{3}\pi$.

4. Find the moment of inertia of a right circular cylinder of height h, radius of whose base is a. Ans. $\frac{1}{2}\pi$ h a^4

5. Find the moment of inertia for the area bounded by $y=5x-x^2$ and $y=2x$ about x-axis. Ans. $\frac{37}{25}$.

6. Show that the moment of inertia of an elliptic disc of axes $2a$, $2b$ about the y axis is $M\frac{a^2}{4}$, M being the mass of the disc.

7. Show that the moment of inertia of a hollow sphere of radius a and mass M about a diameter is $M.\frac{2a^2}{3}$.

8. If AY be drawn perpendicular to the plane of the triangle ABC, show that $M.I.$ of the triangular area ABC about AY is $\frac{M}{12}(3b^2+3c^2-a^2)$, where M is the mass of the triangular area.

9. If a and b be the external and internal radii of a hollow sphere, show that its $M.I.$ about a diameter is $\frac{2M}{3}\frac{a^5-b^5}{a^3-b^3}$.

10. Show that the moment of inertia of a right circular cone of height h about a slant side is $\frac{3Ma^2}{20}.\frac{6h^2+a^2}{h^2+a^2}$

where a is the radius of the base of the cone.

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